Weak Factorisation Systems in the Effective Topos

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis we present a new model of Martin-Löf type theory with identity types in the effective topos. Using the homotopical approach to type theory, the model is induced from a Quillen model category structure on a full subcategory of the effective topos. To aid the construction we introduce a general method of obtaining model category structures on a full subcategory of an elementary topos, by starting from an *interval object* I and restricting our attention to *fibrant* objects, utilizing the notion of fibrancy similar to the one that Cisinksi employed [10] for constructing a model category structure on a Grothendieck topos with an interval object.

We apply this general method to the effective topos $\mathcal{E}ff$. Following Van Oosten [28] we take the interval object to be $I = \nabla(2)$, and derive a model structure on the subcategory $\mathcal{E}ff_f$ of fibrant objects. This Quillen model category structure gives rise to a model of type theory in which the identity type for a type X is represented by X^I. It follows that the resulting model supports functional extensionality.

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Contents

Introduction

Homotopy Type Theory (HoTT for short) [40] is a development in mathematical logic linking dependent type theory and homotopy theory, in which types are interpreted as topological spaces, and the identity type is interpreted as a path space. The aim of the HoTT program is to establish a further connection between algebraic topology (specifically, homotopy theory) and logic. On the one hand, homotopy type theory is concerned with developing a formal synthetic theory of homotopy types; on the other hand it is concerned with providing a homotopical interpretation of type theory. This thesis is primarily motivated by the latter aim.

In order to provide a model for dependent type theory with the identity type we make use of the framework of model categories. A Quillen model category (or a Quillen model structure on a category) is a setting for abstract homotopy theory; basically it is a category, among other things, equipped with classes of morphisms called *weak equivalences*, *cofibrations* and *fibrations*. Weak equivalences can be formally inverted to obtain a *homotopy category*, which is a generalization of the homotopy category construction for topological spaces or simplicial sets.

Awodey & Warren showed in their seminal work [2] that any Quillen model category gives rise to a model of type theory with intesional identity types, in which types are interpreted as fibrations. Working in the "opposite" direction, Gambino & Garner showed in [16] that a syntactic category associated to dependent type theory with identity types possesses a weak factorisation system (a part of a model category), in which path objects are given using identity types.

There are many examples of Grothendieck topoi carrying model category structure. In fact, Cisinski has shown [10] that there is a general method of constructing a model category structure on a Grothendieck topos in which cofibrations are exactly the monomorphisms. However, Cisinski's construction makes use of the cocompleteness of Grothendieck topoi and, therefore, cannot be applied to all elementary topoi. A notable example of an elementary topos that is not cocomplete is the effective topos.

The effective topos of Hyland [22] is a unique object in topos theory. While not being a Grothendieck topos, it serves as a universe for constructive/computable mathematics. The internal logic of $\mathcal{E}ff$ is the "realizability logic". Which dates back to the seminal work of Kleene [26], in which he showed how computable (at that time, partial recursive) functions can be used to give a strong constructive interpretation of arithmetic. Unlike constructive systems like $\mathbf{HA}^\omega$, this interpretation is incompatible with classical logic. For instance, *Church's Thesis* holds internally in $\mathcal{E}ff$, meaning that $\mathcal{E}ff$ can prove that every function from natural numbers to natural numbers is computable (this fact is also observed "externally").

The main topic of this thesis is the study of abstract homotopy in the effective topos $\mathcal{E}ff$, with the aim of obtaining a model of intensional type theory which supports functional extensionality. While the existence of a non-trivial model category structure on the effective topos is still an open question, we managed to establish a model category structure on a full subcategory of $\mathcal{E}ff$. Namely, on the subcategory of *fibrant objects*.

Following Gambino & Garner [17] and Cisinski [10] we work in a topos $\mathcal{C}$ with an interval object I (with connections); given a class $\mathcal{I}$ of maps in $\mathcal{C}$, we define an *$\mathcal{I}$-fibration* to be a map that has a right lifting property against the inclusions of the form $(I \times A) \cup (\{e\} \times B) \hookrightarrow I \times B$ for $e = 0, 1$ and $A \hookrightarrow B \in \mathcal{I}$. Using the interval object one can define a homotopy H to be a map $I \times X \to Y$ such that the source and the target of H are the restrictions of H to $\{0\} \times X \to Y$ and $\{1\} \times X \to Y$, respectively. Then we say that a map $f : X \to Y$ is a homotopy equivalence if there is a map $g : Y \to X$ going in the opposite direction, such that $f \circ g$ and $g \circ f$ are identities up to homotopy.

Fixing $\mathcal{I} = Mono$ and using those notions of a homotopy equivalence and of a fibration we define a model structure on the subcategory $\mathcal{C}_f \hookrightarrow \mathcal{C}$ of fibrant objects (objects X for which the unique map $X \to 1$ is an $\mathcal{I}$-fibration). In the resulting model category, cofibrations are monomorphisms, weak equivalences are homotopy equivalences, and fibrations are $\mathcal{I}$-fibrations. For constructing the model structure, we make use of the so-called *strong* homotopy equivalences; that is, homotopy equivalences for which there is an additional "coherence" requirement on homotopies. They closely correspond to adjoint equivalences in homotopy type theory, except we require that the adjunction condition is satisfied "on the nose". We show that if we restrict our attention to $\mathcal{C}_f$, both acyclic cofibrations and acyclic fibrations can be endowed with a structure of a strong homotopy equivalence (in the strict sense).

In order to apply this method to the effective topos we develop upon the ideas of Van Oosten [28] on the notion of homotopy in $\mathcal{E}ff$. Van Oosten showed that objects in $\mathcal{E}ff$ can be seen as "spaces", and for each object one can construct a path object, playing a role of a "path space". From that one can derive many appropriate notions, for instance notions of path, homotopy and path contraction. The path object is set up in such a way that the familiar discrete objects of the effective topos (quotients of subobjects of the natural numbers object $\mathbf{N}$) correspond to "discrete spaces", and taking the discrete reflection corresponds to taking the set of path components.

This construction yields a path object category in the sense of Van den Berg & Garner [6], but not a Quillen model category. In the resulting model of type theory any object can be interpreted as a type, but the model does not support functional extensionality.

We take the interval object I to be $\nabla 2$. Then a path $p : I \to X$ in an assembly X with the realizability relation E_X consists of two points $p(0), p(1)$, that share a realizer $n \in E_X(p(0)) \cap E_X(p(1))$. This choice of an interval object yields a model category structure, as discussed. In this model structure the notions of a contractible object (the map $X \to 1$ is an acyclic fibration) and an injective object coincide. One can then show that contractible objects are uniform (covered by ∇X for some X), and uniform fibrant objects are contractible. Furthermore, uniform assemblies are contractible, given that one can do a bit of classical reasoning in the ambient set theory. Specifically, the interval object I is contractible iff **Set** satisfies the weak form of the restricted law of excluded middle. Furthermore, the identity type for the object X is interpreted as X^I.

Thus, examples of types in our model include: $\mathbf{N}$, Ω, all the power objects & all the finite types. The types are also closed under products, exponentiation, Π and Σ types.

The model presented in this thesis is different from that of Van Oosten [28], in particular our model supports functional extensionality (as $(X^Y)^I \simeq (X^I)^Y$ is true in any Cartesian closed category). However, our models has similarities with that of Van Oosten. For instance, we were able to show that for a fibrant X, the path object X^I is homotopic to the path object PX in the sense of Van Oosten.

Structure of the thesis. The thesis is divided into two parts: the first one being more abstract and dealing with general categorical frameworks; the second part dedicated to the effective topos and the applications of the results from the first part.

- *Chapter 1.* The first chapter contains preliminaries concerning category theory and model categories.

- *Chapter 2.* Next chapter is devoted to the study of $\mathcal{I}$-fibrations and related notions of homotopy. In this chapter we recall notions of a trivial uniform $\mathcal{I}$-fibration and a uniform $\mathcal{I}$-fibration, homotopy and strong homotopy equivalence. We also prove some standard homotopy-theoretic results in this setting.

- *Chapter 3.* In this chapter we prove the following statement: there exists a model category structure on a subcategory of $\mathcal{I}$-fibrant objects $\mathcal{C}_f \hookrightarrow \mathcal{C}$, where the total category $\mathcal{C}$ is a topos, and $\mathcal{I}$ is the class of all monomorphisms. The required (acyclic cofibration, fibration) factorisation system exists on "weaker" categories then topoi and for $\mathcal{I}$ not containing all monos.

- *Chapter 4.* In the fourth chapter we recall the definition of the effective topos $\mathcal{E}ff$, interpretation of logic in the effective topos and various standard classes of objects and maps. In this section we also recall Van Oosten's path object construction and discuss its relation to discrete reflection.

- *Chapter 5.* This chapter is involved with the study of the model structure on the category $\mathcal{E}ff_f$ of fibrant objects of the effective topos. This model structure is constructed using the method presented in Chapter 3. We manage to draw connections between fibrant objects and discrete objects; between contractible objects and uniform objects. We also give a concrete description of the homotopy category of fibrant assemblies. Finally, we compare our results to the construction of Van Oosten.

- *Chapter 6.* In the final chapter we show how the model category structure on $\mathcal{E}ff_f$ gives rise to the model of type theory with identity types, Π and Σ types. We also explain how the model supports functional extensionality.

- *Appendix.* The appendix contains a proof of a folklore theorem relating Π-types and the Frobenius condition.

Thus, the original contributions of this thesis are: a general method of constructing Quillen model structures on full subcategories of topoi; a study of such a model structure on a subcategory of the effective topos in realizability terms; a new model of dependent type theory with functional extensionality in the effective topos.

Notation

Category theory. Given a category $\mathcal{C}$, we write 0 or $\emptyset$ for an initial object in $\mathcal{C}$ (if it exists), and 1 for the terminal object. For any object $X \in \mathcal{C}$ we denote the unique maps $0 \to X$ and $X \to 1$ as $\bot_X$ and $!_X$, respectively. If $\mathcal{C}$ is cartesian closed, we write $\overline{f}$ for a transpose of f along the $A \times (-) \dashv (-)^A$ adjunction. By ev we denote the evaluation map $Y \times X^Y \to X$.

Recursion theory. Given natural numbers n, m we write $\{n\}(m)$ or $n \cdot m$ for *Kleene application*. That is, $\{n\}(m) = r$ holds if the Turing machine with the Gödel number n terminates on the input m (written as $\{n\}(m)\downarrow$) with the result r; otherwise $\{n\}(m)$ is undefined. By $\{n\}(-)$ we denote a partial function computed by the Turing machine with the Gödel number n.

Throughout the thesis we make liberal use of Gödel encoding and various manipulations of the codes. We use the notation $\langle -, - \rangle$ for primitive recursive pairing, and p_1, p_2 for primitive recursive projections. We write recursively encoded sequences like $\langle a, b, c \rangle$.

Because Kleene application $\cdot$ on natural numbers is "functionally complete", we are justified in using λ-notation for writing recursive functions.

We shall freely use the "pattern matching" notation for anonymous functions. For instance, by a term $\lambda \langle a, b \rangle . t(a, b)$ we mean $\lambda x . t(\mathsf{p}_1 x, \mathsf{p}_2 x)$, and similarly for finite sequences.

Part I

Model category structure on a full subcategory of a topos

Chapter 1

Model categories and adjunctions

In this chapter we recall category-theoretic preliminaries that will play a paramount role in this work.

The chapter is structured as follows. First, we recall the theory of weak factorisation systems and Quillen model categories. Secondly, we will discuss categories of orthogonal maps, and how they commute with adjunctions via the Leibniz construction (in the sense of [36] and [35, Chapter 11]).

1.1 Factorisation systems and model categories

In this section we recall the basic notions of factorisation systems and Quillen model categories [33]. A Quillen model category (or a model category for short), is a framework for abstract homotopy theory, and it generalizes a wide range of mathematical settings. See [18, 14] for more information on model categories and their relation to algebraic topology.

1.1.1 Factorisation systems

Perhaps, one of the main categorical ingredients used in this thesis is the notion of *(weak) orthogonality* / weak lifting property. Consider a category $\mathcal{C}$. We say that a map f has the (weak) left lifting property against a map g, written as $f \pitchfork g$, if any commutative diagram of the form

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow{v} & \cdot \end{array}$$

has a diagonal *filler* – a map h, such that $g \circ h = v$ and $h \circ f = u$. Equivalently, we may say that g has the right lifting property against f. For a given map f we can consider the class of maps that have the right lifting property against f, and for a map g we can consider the class of maps that have the left lifting property against g:

$$\begin{cases} f^{\pitchfork} := \{g \in \mathcal{C}^{\to} \mid f \pitchfork g\} \\ {}^{\pitchfork}g := \{f \in \mathcal{C}^{\to} \mid f \pitchfork g\} \end{cases}$$

If X is a class of maps, we write $X^{\pitchfork} := \bigcap_{f \in X} f^{\pitchfork}$, and similarly for ${}^{\pitchfork}X$.

Definition 1.1. A *weak factorisation system* on a category $\mathcal{C}$ is a pair of classes of maps $(\mathcal{L}, \mathcal{R})$ satisfying the following axioms:

- Any map $f : X \to Y$ in $\mathcal{C}$ can be factored as $f = pi$, where $i \in \mathcal{L}$ and $p \in \mathcal{R}$;
- $\mathcal{L}^{\pitchfork} = \mathcal{R}$ and $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$.

1.1.2 Model categories

We assume to work with a closed category $\mathcal{C}$.

Definition 1.2. A *model category* structure on $\mathcal{C}$, consists of three classes of morphisms $\mathcal{W}, \mathcal{F}ib, \mathcal{C}of \subseteq \mathcal{C}^{\to}$, such that both $(\mathcal{F}ib \cap \mathcal{W}, \mathcal{C}of)$ and $(\mathcal{F}ib, \mathcal{C}of \cap \mathcal{W})$ are weak factorisation systems, and the class $\mathcal{W}$ satisfies the *two-out-of-three* property. That is, if f and g are composable maps, and if any two maps from $\{f, g, gf\}$ are in $\mathcal{W}$, then all three of them are in $\mathcal{W}$.

The maps in $\mathcal{W}$ are called *weak equivalences*, the maps in $\mathcal{F}ib$ and $\mathcal{C}of$ are called *fibrations* and *cofibrations*, respectively. A map in $\mathcal{F}ib \cap \mathcal{W}$ is called an *acyclic fibration* and a map in $\mathcal{C}of \cap \mathcal{W}$ is called an *acyclic cofibration*.

If the underlying model category structure for $\mathcal{C}$ is evident from the context, we write that $\mathcal{C}$ is a model category.

Definition 1.3. A model category $\mathcal{C}$ is *right proper* if pullbacks of weak equivalences along fibrations exist, and are weak equivalences.

A model category $\mathcal{C}$ is *left proper* if pushouts of weak equivalences along cofibrations exist, and are weak equivalences.

Definition 1.4. In a model category $\mathcal{C}$ with a terminal object 1 and an initial object 0, an object X is said to be *fibrant* if the unique map $X \to 1$ is a fibration. An object X is said to be *cofibrant* if the unique map $0 \to X$ is a cofibration.

Example 1.5 (Groupoids)**.** We say that a map $F : C \to D$ of groupoids (that is, a functor) is an *isofibration* if for any object $c \in C$ and a map $k : d' \to F(c)$ in D, there exists a map $l : c' \to c$, such that $F(l) = k$.

Then there is a model category structure on the category **Gpd** of groupoids, where weak equivalences are equivalences of categories, fibrations are isofibrations, and cofibrations are functors that are injective on objects. See [34] for details.

Example 1.6 (Simplicial sets)**.** The category $SSet$ of simplicial sets possesses a model category structure in which cofibrations are monomorphisms, fibrations are Kan fibrations, and weak equivalences are weak homotopy equivalences (maps that induce isomorphisms of fundamental groups). See [25] for details.

Proposition 1.7 ([18, Theorem 7.6.10])**.** *Given a model category $\mathcal{C}$ and an object $X \in \mathcal{C}$, the slice category $\mathcal{C}/X$ is a model category as well, in which a morphism is a fibration/cofibration/weak equivalence iff the underlying map in $\mathcal{C}$ is a fibration/cofibration/weak equivalence.*

1.2 Adjunction situations and Leibniz construction

In this section we would like to recall some of the basic notions that we borrowed from the framework of Gambino & Sattler [17].

Suppose we have an adjunction $\mathcal{E} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{F}$ ($F \dashv G$). Then the adjunction lifts to the adjunction between the arrow categories $\mathcal{E}^{\to} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{F}^{\to}$ ($F \dashv G$), in which lifted functors are defined pointwise. For instance, the image of f under F is $F(f)$, and the image of a commutative square $f \to g$ under F is obtained component-wise.

Proposition 1.8. $\mathcal{E}^{\to} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{F}^{\to}$ ($F \dashv G$) *is an adjunction*

Proof. Given a square as in the picture below on the left, we have to produce a square as in the picture below on the right.

$$\begin{array}{ccc} FX & \xrightarrow{h} & A \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle g} \\ FY & \xrightarrow[k]{} & B \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & GA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Gg} \\ Y & \longrightarrow & GB \end{array}$$

The arrow $X \to GA$ is obtained by transposing $h : FX \to A$, and the arrow $Y \to GB$ is obtained by transposing $k : FY \to B$; the commutativity of the square follows by the naturality of the adjunciton.

To see that this *Hom*-set isomorphism is natural, suppose that $\alpha : f' \to f$ is a square in $\mathcal{E}^{\to}$; then it remains to check the commutativity of

$$\begin{array}{ccc} Hom(Ff, g) & \xrightarrow{-\circ F\alpha} & Hom(Ff', g) \\ \downarrow \wr & & \downarrow \wr \\ Hom(f, Gg) & \xrightarrow[-\circ\alpha]{} & Hom(f', Gg) \end{array}$$

A simple diagram chase, emplying the naturality of the original adjunction, should suffice to verify that fact. ☐

An important consequence of this fact is that the orthogonality commutes with adjunctions.

Theorem 1.9 ([17, Proposition 2.8]). *Let* $\mathcal{I} \subseteq \mathcal{E}^{\to}$ *and* $\mathcal{J} \subseteq \mathcal{F}^{\to}$ *be classes of maps, and let* $\mathcal{E}^{\to} \underset{G}{\overset{F}{\underset{\bot}{\rightleftarrows}}} \mathcal{F}^{\to}$ *be an adjunction. The following two conditions are equivalent:*

1. $F(\mathcal{I}) \subseteq {}^{\pitchfork}\mathcal{J}$
2. $\mathcal{I}^{\pitchfork} \supseteq G(\mathcal{J})$

Proof. We prove $(i) \Rightarrow (ii)$, as the other direction is similar. This amounts to constructing fillers for the diagrams of the following form:

$$\begin{array}{ccc} A_i & \xrightarrow{s} & GX \\ \downarrow{\scriptstyle u_i} & & \downarrow{\scriptstyle G(v_j)} \\ B_i & \xrightarrow{t} & GY \end{array}$$

with $u_i \in \mathcal{I}$. By proposition 1.8 the adjunction $F \dashv G$ lifts to the adjunction between the arrow categories. The diagram above can be seen as a morphism in $\mathcal{E}$. Transposing this diagram along the lifted adjunction, we get a diagram of the form

$$\begin{array}{ccc} F(A_i) & \xrightarrow{\bar{s}} & GX \\ \downarrow{\scriptstyle F(u_i)} & & \downarrow{\scriptstyle v_j} \\ F(B_i) & \xrightarrow{\bar{t}} & GY \end{array}$$

By our assumption this diagram has a filler $\psi : F(B_i) \to GX$. Transposing the diagram with a filler back, we get $\overline{\psi} : B_i \to GX$. We can verify that $\overline{\psi}$ is the desired filler by the naturality of the adjunction. ☐

Once we consider a *two-variable adjunction*, the situation is a bit different, as pointwise calculation will not yield an adjunction. However, we can use the Leibniz construction (in the sense of [36]) to lift the two-variable adjunction to the level of categories of maps.

Consider a two-variable adjunction

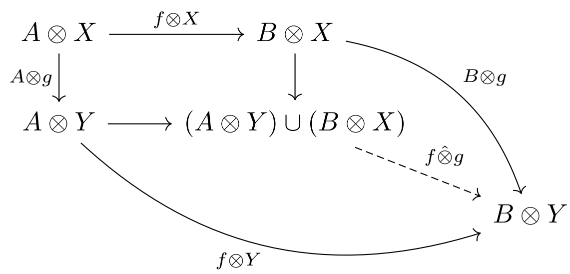
$$- \otimes - \dashv \exp(-,-) \dashv \{-,-\}$$

for a bifunctor $- \otimes - : \mathcal{C} \times \mathcal{D} \to \mathcal{F}$. If $\mathcal{F}$ has pushouts and $\mathcal{C}$ and $\mathcal{D}$ have pullbacks, there is an induced adjunction

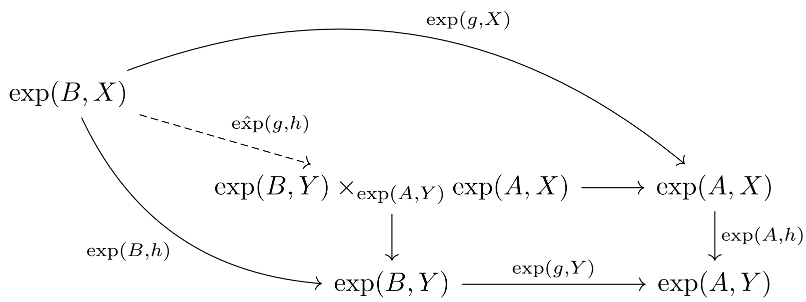
$$-\hat{\otimes}- \dashv \hat{\exp}(-,-) \dashv \{-\hat{,}-\}$$

between the arrow categories given by the so-called *Leibniz construction.* In the terminology [35], the induced bifunctors are given by the *pushout-product*, *pullback-cotensor*, and *pullback-hom* constructions:

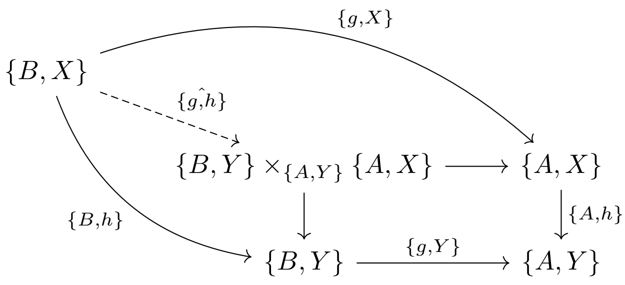
Definition 1.10 (Leibniz product / pushout-product)**.** Given a functor $-\otimes- : \mathcal{C}\times\mathcal{D} \to \mathcal{F}$, define a functor $-\hat{\otimes}- : \mathcal{C}^{\to} \times \mathcal{D}^{\to} \to \mathcal{F}^{\to}$ as follows: given $f : A \to B \in \mathcal{C}^{\to}$ and $g : X \to Y \in \mathcal{D}^{\to}$, $f\hat{\otimes}g$ is obtained from a pushout



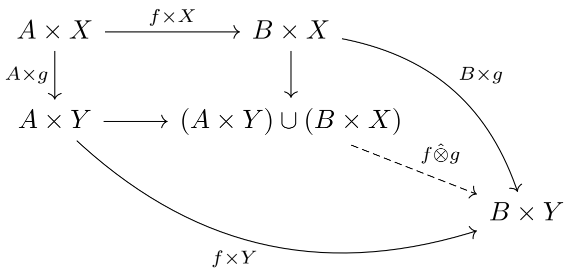
Definition 1.11 (Pullback exponent / pullback-hom)**.** Given a functor $\exp : \mathcal{D}^{op} \times \mathcal{F} \to \mathcal{C}$, define a functor $\hat{\exp} : (\mathcal{D}^{op})^{\to} \times \mathcal{F}^{\to} \to \mathcal{C}^{\to}$ as follows: given $g : A \to B \in \mathcal{D}^{\to}$ and $h : X \to Y \in \mathcal{F}^{\to}$, $\hat{\exp}(g, h)$ is obtained from a pullback

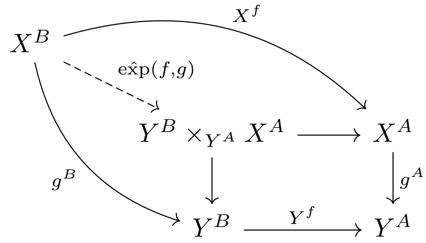


Definition 1.12 (Pullback-cotensor)**.** The induced bifunctor $\{-\hat{,}-\}$ is obtained similarly from a pullback:



If we start with a cartesian product/exponent adjunction $(-) \times (-) \dashv (-)^{(-)}$, then $\exp(-, -) = \{-, -\}$ and we get the following constructions that we will use throughout the thesis:

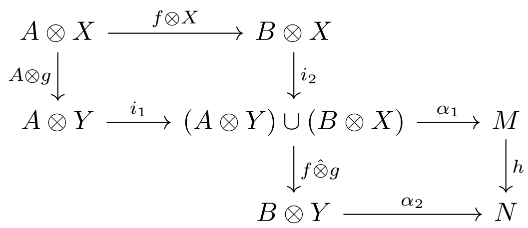




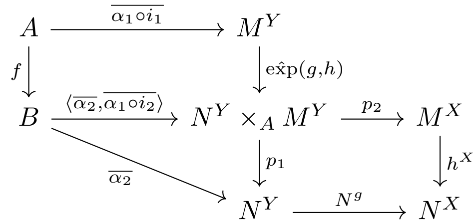
Proposition 1.13. *The construction above induces a two-variable adjunction* $-\hat{\otimes}- \dashv \hat{\exp}(-,-) \dashv \{-\hat{,}-\}$.

Proof. This amounts to showing $Hom_{\mathcal{F}^\to}(f\hat{\otimes}g, h) \simeq Hom_{\mathcal{C}^\to}(f, \hat{\exp}(g,h)) \simeq Hom_{\mathcal{D}^\to}(g, \{f \hat{,} h\})$. A proof of this is a laborious diagram chase. We will only construct a mapping $Hom_{\mathcal{F}^\to}(f\hat{\otimes}g, h) \to Hom_{\mathcal{C}^\to}(f, \hat{\exp}(g,h))$. We use the exponentiation notation, i.e. we write X^A for $\exp(A, X)$ and use $\overline{}$ to denote the isomorphism $Hom_{\mathcal{F}}(A \otimes B, C) \simeq Hom_{\mathcal{C}}(A, \exp(B, C))$.

Suppose we have a commutative square $(\alpha_1, \alpha_2) : f\hat{\otimes}g \to h$:



Then we construct the map $\beta : f \to \hat{\exp}(g, h)$ as follows.



(1.1)

The first component of β is $\overline{\alpha_1 \circ i_1} : A \to M^Y$. To construct the second component of β we use the universal property of the pullback to obtain $\langle \overline{\alpha_2}, \overline{\alpha_1 \circ i_2} \rangle : B \to N^Y \times_{N^X} M^X$. This map is well-defined, because

$$
\begin{aligned}
h^X \circ \overline{\alpha_1 \circ i_2} &= \\
\overline{h \circ \alpha_1 \circ i_2} &= \\
\overline{\alpha_2 \circ (B \otimes g)} &= N^g \circ \overline{\alpha_2}
\end{aligned}
$$

To verify that diagram (1.1) commutes, it suffices to verify the commutativity up to post-composition with p_1 and p_2.

1.

$$
\begin{aligned}
p_2 \circ \hat{\exp}(g,h) \circ \overline{\alpha_1 \circ i_1} &= \text{(by the definition of } \hat{\exp}(g,h)) \\
M^g \circ \overline{\alpha_1 \circ i_1} &= \text{(by the naturality of the adjunction)} \\
\overline{\alpha_1 \circ i_1 \circ (A \otimes g)} &= \\
\overline{\alpha_1 \circ i_2 \circ (f \otimes X)} &= \text{(by the naturality of the adjunction)} \\
\overline{\alpha_1 \circ i_2} \circ f &= p_2 \circ \langle \overline{\alpha_2}, \overline{\alpha_1 \circ i_2} \rangle \circ f
\end{aligned}
$$

2. And, similarly:

$$
\begin{aligned}
p_1 \hat{\exp}(g,h) \circ \overline{\alpha_1 \circ i_1} &= \text{ (by the definition of } \hat{\exp}(g,h)) \\
h^Y \circ \overline{\alpha_1 \circ i_1} &= \\
\overline{h \circ \alpha_1 \circ i_1} &= \\
\overline{\alpha_2 \circ f \hat{\otimes} g \circ i_1} &= \\
\overline{\alpha_2 \circ (f \otimes Y)} &= \\
\overline{\alpha_2} \circ f &= p_1 \circ \langle \overline{\alpha_2}, \overline{\alpha_1 \circ i_2} \rangle \circ f
\end{aligned}
$$

□

The following corollary is used in the proofs throughout the thesis.

Corollary 1.14. *A map u has the left lifting property against* $\hat{\exp}(f,g)$ *if an only if the map g has the right lifting property against the map $u \hat{\otimes} f$.*

$$
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \hat{\exp}(f,g)} \\
\cdot & \longrightarrow & \cdot
\end{array}
\qquad
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
{\scriptstyle u\hat{\otimes} f}\downarrow & & \downarrow{\scriptstyle g} \\
\cdot & \longrightarrow & \cdot
\end{array}
$$

Notes

In this chapter we recalled some preliminaries from category theory and abstract homotopy theory, which will be used throughout this thesis. Specifically, we covered weak factorisation systems and Quillen model categories, which are widely used in homotopy theory. One of the main ingredients in weak factorisation system is the notion of orthogonality. In this chapter we have seen how orthogonality factors through adjunctions, a trick that will be widely used in many proofs below. This is due to the fact that an adjunction between categories $\mathcal{C}$ and $\mathcal{D}$ lifts to an adjunction between the categories of maps $\mathcal{C}^{\rightarrow}$ and $\mathcal{D}^{\rightarrow}$. Similar result can be obtained for two-variable adjunctions, but in order to lift a two-variable adjunction to the level of categories of maps, one has to employ the Leibniz construction.

In the next chapter we will employ the Leibniz construction once again to define $\mathcal{I}$-fibrations; in order to prove several properties about them we will use the trick of transporting the lifting problems along adjunctions.

Chapter 2

$\mathcal{I}$-fibrations and homotopy equivalences

In this chapter we study a notion of a *uniform $\mathcal{I}$-fibration*, which arises in the framework of Gambino and Sattler [17], and is inspired by the work of Cisinski [10]. In this framework we are working in a Cartesian closed category with an *interval object* $1 \rightrightarrows I$, which is used to define the notions of *homotopy* and *(trivial) uniform fibrations*. Intuitively, we start with a certain class of maps $\mathcal{I}$, and define uniform fibrations to be such maps that have the right lifting property with regard to "inclusions" $(\{e\} \times B) \cup (I \times A) \hookrightarrow I \times B$ for $A \hookrightarrow B \in \mathcal{I}$, $e = 0, 1$.

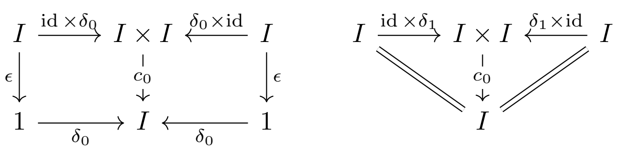
The work in this chapter is devoted to establishing standard homotopy-theoretic results in this framework inspired by the work of Gambino & Sattler. Although we haven't been able to locate some of the results in this chapter in the literature (at least not in the specific setting under the specific formulations), we do not wish to take credit for the originality, as most of those results are either similar to the standard facts in classical topology and homotopy theory, or follow from the propositions in [17] and [10].

The chapter is structured as follows. In section 2.1 we recall the definitions of an interval object, $\mathcal{I}$-fibration and a trivial $\mathcal{I}$-fibration starting from a class of maps $\mathcal{I}$. We also state and prove some closure properties of $\mathcal{I}$-fibrations and fibrant objects; for instance, we show how under certain conditions, $\mathcal{I}$-fibrations satisfy a "box filling" property that can be used to reason pictorially. In section 2.2 we discuss the notion of homotopy and different types of homotopy equivalences. We show that those notions are "well-defined" with regard to fibrant-cofibrant objects. We differentiate between regular homotopy equivalences and strong homotopy equivalences (in which we require the given homotopies to satisfy a certain coherence condition). We show that an $\mathcal{I}$-fibration or a morphism from $\mathcal{I}$ that is a homotopy equivalence can be endowed with a structure of a strong homotopy equivalence.

2.1 Interval object and $\mathcal{I}$-fibrations

Suppose we have a Cartesian closed category $\mathcal{C}$. We say that $\mathcal{C}$ possesses an *interval object* if there is an object $I \in \mathcal{C}$, with maps $\delta_0, \delta_1 : 1 \to I$ which are called *endpoint inclusions*. In a Cartesian category, the interval I also has *contractions*, that is a map $\epsilon : I \to 1$.

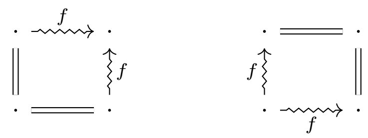
Connections. An interval object I is said to have *connections* if there are maps $c_0, c_1 : I \times I \to I$ such that the diagrams



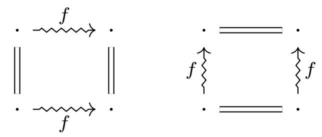
and

$$\begin{array}{ccccc} I & \xrightarrow{\mathrm{id}\times\delta_1} & I\times I & \xleftarrow{\delta_1\times\mathrm{id}} & I \\ {\scriptstyle\epsilon}\downarrow & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle\epsilon} \\ 1 & \xrightarrow[\delta_1]{} & I & \xleftarrow[\delta_1]{} & 1 \end{array} \qquad \begin{array}{ccccc} I & \xrightarrow{\mathrm{id}\times\delta_0} & I\times I & \xleftarrow{\delta_0\times\mathrm{id}} & I \\ & \searrow\!\!\!= & \downarrow{\scriptstyle c_1} & =\!\!\!\swarrow & \\ & & I & & \end{array}$$

commute. Intuitively, an interval $f : I \to X$ in an object X can be seen as a degenerate square in several different ways. Squares $f \circ c_0$ and $f \circ c_1$ are shown in the diagram below on the left and on the right, respectively.



One can also view an interval $f : I \to X$ as "squares"



via maps

$$I \times I \xrightarrow{\pi_1} I \xrightarrow{f} X \qquad I \times I \xrightarrow{\pi_2} I \xrightarrow{f} X$$

Thus, the presence of connections allows us to view any path/1-cell $f : I \to X$ as a square/2-cell in all 4 possible ways.

For the remainder of this chapter, we fix a Cartesian closed category $\mathcal{C}$ with finite limits, pushouts, and an interval object I. We shall also assume that the functor $I \times (-)$ has a right adjoint $(-)^I$; intuitively, X^I is an object of paths in X. Then the endpoint inclusion maps induce maps $X^{\delta_0} : X^I \to X$, $X^{\delta_1} : X^I \to X$, which we call *source* and *target* maps, respectively, and denote as s and t, when unambiguous. We also have a *reflexivity map* $r = X^\epsilon : X \to X^I$, which intuitively sends x to the constant path at x.

Example 2.1 (Groupoids)**.** Consider a groupoid $\mathbf{I}$ which consists of two objects $0, 1$ and two non-identity arrows $\iota : 0 \to 1, \iota^{-1} : 1 \to 0$, such that $\iota \circ \iota^{-1} = \mathrm{id}_1$ and $\iota^{-1} \circ \iota = \mathrm{id}_0$. This groupoids is an interval object in the category **Gpd** of groupoids.

The endpoint inclusions $\delta_i : 1 \to \mathbf{I}$ are functors that select out $i \in \mathbf{I}$. The connection structure is provided via functors $min, max : \mathbf{I} \times \mathbf{I} \to \mathbf{I}$ as described below

$$\begin{cases} min(0, j) = min(i, 0) = 0 \\ min(1, j) = min(j, 1) = j \\ min(\iota, \iota) = \iota \\ min(\mathrm{id}_i, f) = min(g, \mathrm{id}_i) = \mathrm{id}_i \end{cases}$$

and similarly for max.

Example 2.2 (Lawvere interval)**.** Given a topos $\mathcal{E}$, the subobject classifier Ω can play a role of the interval object. The endpoints δ_0, δ_1 are given by $\bot : 1 \to \Omega$ and $\top : 1 \to \Omega$, respectively. The connection structure is given by $\wedge : \Omega \times \Omega \to \Omega$ and $\vee : \Omega \times \Omega \to \Omega$, respectively.

Uniform $\mathcal{I}$-fibrations. Our goal is the next chapter is to construct a model structure starting from the class of cofibrations $\mathcal{I}$, which is a subclass of monomorphisms in the category. Using the interval object I in a category $\mathcal{C}$ and the Leibniz construction (definition 1.11), we will define a notion of a $\mathcal{I}$-fibration, which will serve as the class of fibrations in the model structure. Of course, the class of fibrations should contain exactly those maps that have a right lifting property against cofibrations which are weak equivalences. As we will see, fibrations can also be described by a lifting property against a smaller class, not involving any notions of equivalence. Specifically, an $\mathcal{I}$-fibration is a map which has a right lifting property against all the maps of the form $(\{e\} \times B) \cup (I \times A) \to I \times B$, which are obtained via the Leibniz construction from a map $u : A \to B \in \mathcal{I}$.

Formally, given a class of maps $\mathcal{I} \subseteq \mathcal{C}^{\to}$, we construct a class of maps $\mathcal{I}_{\otimes} \subseteq \mathcal{C}^{\to}$. The class $\mathcal{I}_{\otimes}$ contains maps of the form $(\{k\} \times B) \cup (I \times A) \xrightarrow{\delta_k \hat{\otimes} u} I \times B$, for $u : A \to B$ and $k = 0, 1$.

Definition 2.3. A uniform $\mathcal{I}$-fibration is a map that has a right lifting property against a map from $\mathcal{I}_{\otimes}$. That is, $\mathcal{I}\text{-}\mathcal{F}ib := (\mathcal{I}_{\otimes})^{\pitchfork}$.

Similarly, we define trivial $\mathcal{I}$-fibrations; they will be acyclic fibrations in the resulting model structure.

Definition 2.4. A uniform trivial $\mathcal{I}$-fibration is a map that has a right lifting property against a map from $\mathcal{I}$. That is, $Triv\mathcal{F}ib := \mathcal{I}^{\pitchfork}$.

An object X is said to be ($\mathcal{I}$-)*fibrant* if the unique map $X \to 1$ is a uniform fibration; likewise, an object X is said to be *trivially fibrant* if the unique map $X \to 1$ is a uniform trivial fibration. If an initial map $0 \to Y$ is in $\mathcal{I}$, then we say that Y is *cofibrant*.

Now we are going to prove some useful propositions about uniform $\mathcal{I}$-fibrations and $\mathcal{I}$-fibrant objects.

2.1.1 $\mathcal{I}$-fibrations and filling conditions

For propositions in this section we assume that the class of maps $\mathcal{I} \subseteq \mathcal{C}^{\to}$

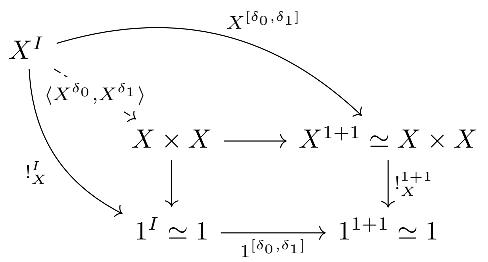
- contains a map $[\delta_0, \delta_1] : 1 + 1 \to I$;
- contains a map $\emptyset \to 1$;
- is closed under Leibniz product.

This is the case if, e.g. $\mathcal{I}$ is the class of monomorphisms. As usual, we require $\mathcal{C}$ to be a Cartesian closed category with pullbacks and pushouts.

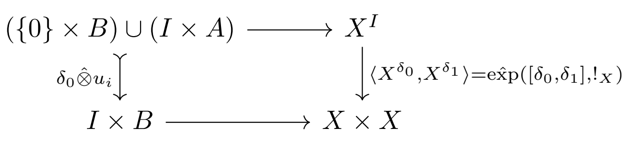
We would like to show how fibrations and fibrant object satisfy a "box filling" property, stating that an "open box" or an "open square" can be filled. This property allows for formal pictorial reasoning that will come in handy in the next chapter. However, first we need an auxiliary proposition.

Proposition 2.5. *Given a fibrant object X, the source/target map $\langle X^{\delta_0}, X^{\delta_1} \rangle = \langle s, t \rangle : X^I \to X \times X$ is also a fibration. Furthermore, if X is trivially fibrant, then $\langle s, t \rangle$ is a trivial fibration.*

Proof. We will prove the first part of this proposition, the proof for the second part is similar. First note that the map $\langle s, t \rangle$ can be expressed as a pullback-hom $\hat{\exp}([\delta_0, \delta_1], !_X)$:



By corollary 1.14, the problem of finding a filler for a diagram of the form



(2.1)

for a map $u_i : A \to B$ reduces to the problem of finding a filler for the diagram below.

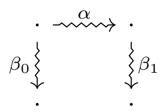
$$\begin{array}{ccc} (\{0\} \times I \times B) \cup (I \times I \times A) \cup (I \times B + I \times B) & \longrightarrow & X \\ {\scriptstyle \delta_0 \hat{\otimes} [\delta_0, \delta_1] \hat{\otimes} u_i} \big\downarrow & & \big\downarrow {\scriptstyle !_X} \\ I \times I \times B & \longrightarrow & 1 \end{array}$$

However, the diagram above has a diagonal filler because we have assumed that X is fibrant. □

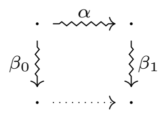
Box filling condition. Suppose that the class of maps $\mathcal{I} \subseteq \mathcal{C}^{\to}$ includes a monomorphism $u : \emptyset \to 1$. Then, as one can check, $\delta_0 \hat{\otimes} u = \delta_0$, and diagram (2.1) becomes

$$\begin{array}{ccc} \{0\} & \longrightarrow & X^I \\ \big\downarrow & & \big\downarrow {\scriptstyle \langle X^{\delta_0}, X^{\delta_1} \rangle} \\ I & \longrightarrow & X \times X \end{array}$$

The map $\{0\} \to X^I$ corresponds to a path α in X, the map $I \to X \times X$ corresponds to a pair of paths (β_0, β_1) in X; the commutativity of the diagram requires the two aforementioned paths to start at the beginning and, respectively, at the end of α, which we can visualize as depicted below.



Then the filler for the diagram (2.1) would give us a path connecting the ends of β_0 and β_1 and a filler for the resulting square



So, in a way, the lifting problems for the diagrams of such form for a fibrant X can be seen as a *square filling condition.* The same argument can be generalized to n-cubes.

Closure properties. Before moving on to the next section, we would like to state some simple but useful facts about $\mathcal{I}$-fibrations, so that we can reference them later in the text.

First of all, we can utilize the proof method of transposing the lifting problems along the $u_i \hat{\otimes} - \dashv \hat{\exp}(u_i, -)$ adjunction to show that fibrations are closed under exponentiation.

Proposition 2.6. *If $p : E \to X$ is a fibration, then so is $\hat{\exp}(u_i, p) : E^Z \to (X^Z \times_{X^Y} E^Y)$ for every $u_i : Y \to Z$. If p is a trivial fibration, then so is $\hat{\exp}(u_i, p)$.*

Proof. Similar to the proof of proposition 2.5; transpose the lifting problem along the $\hat{\otimes} \dashv \hat{\exp}$ adjunction.

□

Remark 2.7. In particular, if $\mathcal{I} \subseteq \mathcal{C}^{\to}$ contains an initial map $u_i : \emptyset \to Y$, then the previous proposition implies that $\hat{\exp}(u_i, p) = p^Y : E^Y \to X^Y$ is a fibration whenever p is. In particular, if X is fibrant then so is X^Y, for any cofibrant object Y.

Secondly, we will state two standard facts from the weak factorisation systems theory. Both of them can be proved by a straightforward diagram chase.

Proposition 2.8. *If $p : E \to X$ is a fibration, and $f : Y \to X$ is a map, then the pullback $f^*(p)$ of p along f is a fibration as well.*

Proposition 2.9. *Fibrations are closed under composition.*

Finally, we would like to show that the class of $\mathcal{I}$-fibrations contains projections. In classical topology, a fibration is a generalized fiber bundle; a fiber bundle in turn can be seen as a generalization of a product of spaces. It is therefore unsurprising that for fibrant objects Cartesian projections are $\mathcal{I}$-fibrations, a fact that we will find useful in a couple of proofs in this thesis. We prove a slightly more general result.

Proposition 2.10. *If $p_1 : E_1 \to X_1$ and $p_2 : E_2 \to X_2$ are fibrations, then so is $p_1 \times p_2 : E_1 \times E_2 \to X_1 \times X_2$.*

Proof. A lifting problem of the form

$$\begin{array}{ccc} (\{0\} \times B) \cup (I \times A) & \xrightarrow{\langle k_1, k_2 \rangle} & E_1 \times E_2 \\ {\scriptstyle \delta_0 \hat{\otimes} u_j} \downarrow & & \downarrow {\scriptstyle p_1 \times p_2} \\ I \times B & \xrightarrow[\langle h_1, h_2 \rangle]{} & X_1 \times X_2 \end{array}$$

reduces to two lifting problems

$$\begin{array}{ccc} (\{0\} \times B) \cup (I \times A) & \xrightarrow{k_1} & E_1 \\ {\scriptstyle \delta_0 \hat{\otimes} u_j} \downarrow & {\scriptstyle g_1} \nearrow & \downarrow {\scriptstyle p_1} \\ I \times B & \xrightarrow[h_1]{} & X_1 \end{array} \qquad \begin{array}{ccc} (\{0\} \times B) \cup (I \times A) & \xrightarrow{k_2} & E_2 \\ {\scriptstyle \delta_0 \hat{\otimes} u_j} \downarrow & {\scriptstyle g_2} \nearrow & \downarrow {\scriptstyle p_2} \\ I \times B & \xrightarrow[h_2]{} & X_2 \end{array}$$

which has solutions g_1 and g_2, respectively, by assumption. Then $\langle g_1, g_2 \rangle$ is the solution to the original lifting problem. □

Remark 2.11. In particular, since id_X (just like any isomorphism) is a fibration, the projections $\pi_1 : X \times Y \to X \simeq X \times 1$ and $\pi_2 : X \times Y \to Y \simeq 1 \times Y$ are fibrations in case Y and X are fibrant, respectively.

In the next section we will examine a notion of a *homotopy* induced by an interval object, and several different notions of homotopy equivalences.

2.2 Homotopy and homotopy equivalences

Given an interval object, we define a *homotopy* between maps in the same hom-set. In this section we prove that this notion is "well-behaved" on fibrant objects. We again require that $\mathcal{I}$ contains the inclusions $[\delta_0, \delta_1] : 1 + 1 \to I$ and $\emptyset \to 1$, and is closed under Leibniz product.

2.2.1 Homotopy relation

Two maps $f, g : A \to B$ are said to be *homotopic* if there is a map $\psi : I \times A \to B$, making the following diagram commute:

$$\begin{array}{ccccc} A & \xrightarrow{\delta_0 \times A} & I \times A & \xleftarrow{\delta_1 \times A} & A \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \psi} & \swarrow {\scriptstyle g} & \\ & & B & & \end{array}$$

Such a map is called a *homotopy*, and we write $\psi : f \sim g$ to signify that. We write $f \sim g$ if the homotopy itself is unspecified.

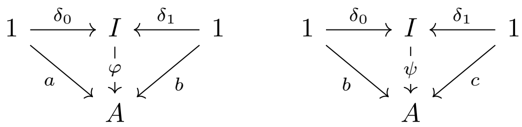
Example 2.12 (Groupoids)**.** Let $\mathcal{G}_1, \mathcal{G}_2$ be groupoids, viewed as maps $1 \to \mathbf{Gpd}$ (modulo size issues). Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are homotopic iff $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic as categories.

The homotopy relation $\sim$ is reflexive (as witnessed by the homotopy $f \circ \pi_2 : I \times A \to B$). In addition, if B is fibrant and A is cofibrant, then $\sim$ is symmetric and transitive on $Hom(A, B)$.

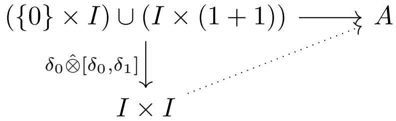
Lemma 2.13. *If A is fibrant, then $\sim$ is an equivalence relation on $Hom(1, A)$.*

Proof. The proof is basically the same as in [10, Lemma 2.11].

Transitivity. Suffices to show that given homotopies



there is a homotopy $a \sim c$. Consider a lifting problem

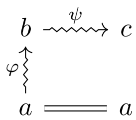


Where the arrow $(\{0\} \times I) \cup (I \times (1+1)) \simeq (\{0\} \times I) \cup (I \times \{0\} + I \times \{1\}) \to A$ is induced by the maps

$$\{0\} \times I \xrightarrow{\varphi} A$$

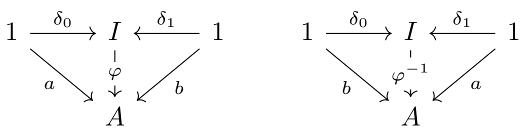
$$(I \times \{0\} + I \times \{1\}) \xrightarrow{[r_a, \psi]} A$$

Where r_a is the transpose of $1 \xrightarrow{a} A \xrightarrow{r} A^I$. Those maps agree on the endpoints, and the induced map corresponds to an "open box"

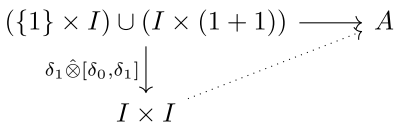


The solution to the lifting problem would provide a filler for that box; in particular we could restrict that filler to a path $I \to A$ with the endpoints a and c. That would be the desired homotopy.

Symmetry. Suffices to show that given a homotopy as on the left below, there is a homotopy as on the right below



Consider a lifting problem



Similar to the previous case, the arrow $(\{1\} \times I) \cup (I \times (1+1)) \simeq (\{1\} \times I) \cup (I \times \{0\} + I \times \{1\})) \to A$ is induced by the arrows

$$\{1\} \times I \xrightarrow{r_b} A$$

$$I \times \{0\} + I \times \{1\} \xrightarrow{[r_b, \varphi]} A$$

which corresponds to an "open box"

$$\begin{array}{ccc} a & \overset{\varphi}{\rightsquigarrow} & b \\ & & \| \\ b & = & b \end{array}$$

a filler for which would provide a path connecting b and a.

$\square$

Lemma 2.14. *If B is fibrant and A is cofibrant, then $\sim$ is an equivalence relation on $Hom(A, B)$.*

Proof. For transitivity, consider homotopies

$$\begin{array}{ccccc} A & \xrightarrow{\delta_0\times A} & I\times A & \xleftarrow{\delta_1\times A} & A \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle\varphi} & \swarrow{\scriptstyle g} & \\ & & B & & \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{\delta_0\times A} & I\times A & \xleftarrow{\delta_1\times A} & A \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle\psi} & \swarrow{\scriptstyle h} & \\ & & B & & \end{array}$$

We can transpose the diagrams to obtain

$$\begin{array}{ccccc} 1 & \xrightarrow{\delta_0} & I & \xleftarrow{\delta_1} & 1 \\ & {\scriptstyle \bar{f}}\searrow & \downarrow{\scriptstyle\bar{\varphi}} & \swarrow{\scriptstyle \bar{g}} & \\ & & B^A & & \end{array} \qquad \begin{array}{ccccc} 1 & \xrightarrow{\delta_0} & I & \xleftarrow{\delta_1} & A \\ & {\scriptstyle \bar{g}}\searrow & \downarrow{\scriptstyle\bar{\psi}} & \swarrow{\scriptstyle \bar{h}} & \\ & & B^A & & \end{array}$$

Then use remark 2.7 and the previous lemma before transposing back. For symmetry the solution is nearly identical. $\square$

Proposition 2.15. *The homotopy relation $\sim$ is a congruence with regard to composition.*

Proof. Given maps $f, g : A \to B$, a homotopy $\psi : f \sim g$, and $h : B \to C$, $k : D \to A$, one can get a homotopy $g \circ k \sim f \circ k$ by precomposing ψ with $I \times k$, and $h \circ f \sim h \circ g$ by postcomposting ψ with h.

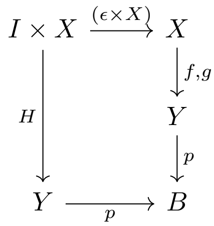
Such composition of ψ with h and k is called *whiskering*. We denote $\psi \circ (I \times k)$ by $\psi.k$ and $h \circ \psi$ by $h.\psi$. $\square$

2.2.2 Fibrewise homotopy

We might want to strengthen a regular notion of homotopy in the following way. Given a homotopy $H : f \sim g$ between $f, g : X \to Y$, where Y lies over a base space B via a fibration $p : Y \to B$, we might want to ask whether the homotopy H "stays" in the same fiber at every "point in time". A useful instance of this general question is as follows. When defining a fundamental group or a groupoid of a space X, we consider homotopies that are constant at endpoints. This amounts to requiring that the homotopy $H : p \sim q$ between paths $p, q : 1 \to X^I$ is constant in the base space $X \times X$ via the fibration $X^I \xrightarrow{\langle s,t \rangle} X \times X$.

Of course, to formulate the general question – whether the homotopy $H : f \sim g$ is constant in the base space – we must require that f and g take image in the same fiber.

Definition 2.16. Given a fibration $p : Y \to B$ and two maps $f, g : X \to Y$ such that $p \circ f = p \circ g$ (that is, f and g lie in the same fiber(s)), we say that f and g are *homotopic over B*, written as $f \sim_B g$, if there is a homotopy $H : f \sim g$, such that the following diagram commutes:



In such case we write $H : f \sim_B g$.

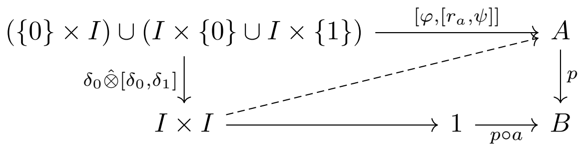
Definition 2.17. Given two maps $f, g : Y \to X^I$, we say that f and g are *homotopic rel endpoints* iff $f \sim_{X \times X} g$. For a homotopy $H : f \sim_{X \times X} g$ we say that H is *constant at endpoints*.

We can prove a result similar to lemma 2.14 for fibrewise homotopies.

Lemma 2.18. *If $p : A \to B$ is a fibration, then $\sim_B$ is an equivalence relation on $Hom(1, A)$.*

Proof. Given a map $a : 1 \to A$, one can verify that $r_a : I \to 1 \xrightarrow{a} A$ is a homotopy $a \sim_B a$ over B. We now establish that $\sim_B$ is also transitive; the symmetry case is similar.

Suppose $a, b, c : 1 \to A$ such that $p \circ a = p \circ b = p \circ c$ and $\varphi : a \sim_B b$, $\psi : b \sim_B c$. Then consider a lifting problem



We can check that the square commutes component-wise, by using the fact that r_a, φ and ψ are homotopies over B. Thus, since p is a fibration, there is a diagonal filler $h : I \times I \to A$. Then take χ to be the composite $\{1\} \times I \hookrightarrow I \times I \xrightarrow{h} A$. By the commutativity of the upper triangle, χ is a homotopy $a \sim c$. In addition,

$$p \circ h = p \circ a \circ (!_{I \times I}) = p \circ c \circ (!_{I \times I})$$

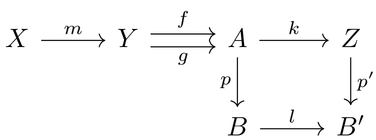
It follows that $p \circ \chi = p \circ a \circ (!_I) = p \circ c \circ (!_I)$, hence χ is a fibrewise homotopy over B. □

Lemma 2.19. *If $p : A \to B$ is a fibration, then $\sim_B$ is an equivalence relation on $Hom(Y, A)$, for a cofibrant Y.*

Proof. This follows from the fact that if $f, g : Y \to A$ are maps such that $p \circ f = p \circ g$, and $\varphi : f \sim_B g$, then by transposition we get $p^Y \circ \overline{f} = p^Y \circ \overline{g}$ and p^Y is a fibration by remark 2.7. In addition $\overline{\varphi} : \overline{f} \sim_{B^Y} \overline{g}$, and thus we can apply the previous lemma. □

We also obtain a result similar to proposition 2.15.

Proposition 2.20. *Let $\varphi : f \sim_B g$ be a fibrewise homotopy between maps $f, g : Y \to A$ over a fibration $p : A \to B$. Let $m : X \to Y$ be a morphism, and let (k, l) be a commutative square between fibrations p and p'.*

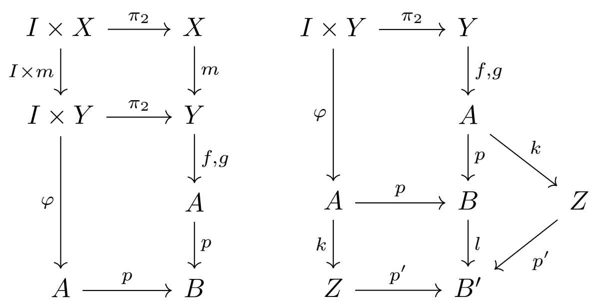


Then

1. $\varphi.m : f \circ m \sim_B g \circ m$

2. $k.\varphi : k \circ f \sim_{B'} k \circ g$

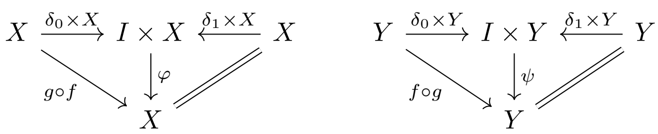
Proof. It is sufficient to check the commutativity of two diagrams below.



□

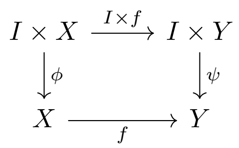
2.2.3 Homotopy equivalences

Definition 2.21. A map $f : X \to Y$ is called a *homotopy equivalence* if there is a map $g : Y \to X$ (called a homotopy inverse of f) and homotopies $\phi : g \circ f \sim \mathrm{id}_X$ and $\psi : f \circ g \sim \mathrm{id}_Y$.



Notice, that we define the homotopies ϕ and ψ to be "right-sided". We focus our attention on fibrant objects, and on maps from cofibrant to fibrant objects. For them the homotopy relation is symmetric; we could have also required the homotopies to be $\phi : \mathrm{id}_X \sim g \circ f$ and $\psi : \mathrm{id}_Y \sim f \circ g$.

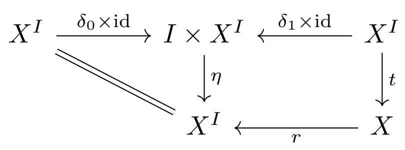
Definition 2.22. A homotopy equivalence f is said to be *strong* if the following diagram commutes.



Example 2.23 (Groupoids)**.** Let $\mathcal{G}_1, \mathcal{G}_2$ be groupoids. There is a homotopy equivalence $F : \mathcal{G}_1 \to \mathcal{G}_2$ iff $\mathcal{G}_1$ and $\mathcal{G}_2$ are equivalent as categories.

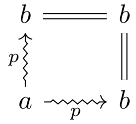
Remark 2.24. In presence of connections, a notable example of a strong homotopy equivalence is the reflexivity map $r = X^\epsilon : X \to X^I$. Its homotopy inverse is $t = X^{\delta_1} : X^I \to X$, which sends the path to its target.

Proof. First of all, $t \circ r = \mathrm{id}_X$ by the definition of the interval with contractions. To show that $r \circ t \sim \mathrm{id}_{X^I}$ one has to provide a filler η for the diagram

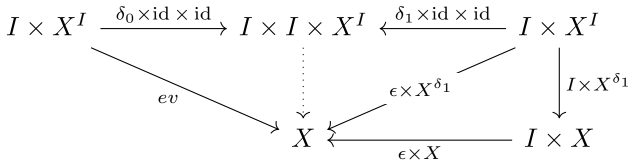


(2.2)

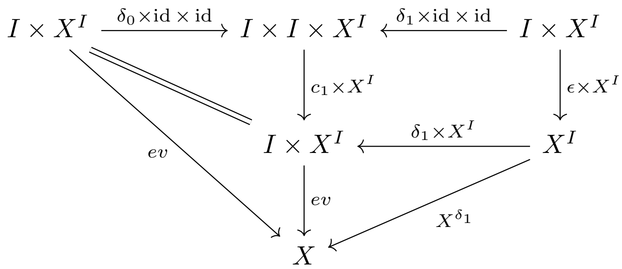
which can be seen as an operation contracting a path onto its endpoint. For instance, one can visualize $\eta(-,p)$ for a path $p : a \rightsquigarrow b$ in X as a square



which suggest the use of connections. By transposing the diagram (2.2), we get



The dotted filler is then given by $ev \circ (c_1 \times X^I)$:



By transposing the diagram back we get η. $\square$

Remark 2.25. Just as $\eta : \mathrm{id}_{X^I} \sim r \circ t$ can be seen a as homotopy contracting a path onto its endpoint, we have a homotopy $\beta : \mathrm{id}_{X^I} \sim r \circ s$ contracting a path onto its starting point. For a fibrant X, β can be obtained by composing $\eta : \mathrm{id}_{X^I} \sim r \circ t$ with $r.\overline{\mathrm{id}}^{-1} : r \circ t \sim r \circ s$.

We obtain the usual definition of the strong deformation retract if we require the second homotopy ψ to be trivial.

Definition 2.26. A map $f : X \to Y$ is a *strong deformation retract* if it is a homotopy equivalence, its homotopy inverse s is also a section: $f \circ s = \mathrm{id}_Y$, and s is a *strong* homotopy equivalence.

2.2.4 Strong and weak homotopy equivalences

Whilst in general not every homotopy equivalence is a strong homotopy equivalence, one can replace a homotopy equivalence that is an $\mathcal{I}$-fibration or a cofibration with a strong homotopy equivalence. Specifically, a cofibration that is a homotopy equivalence can be endowed with a structure of a *strong* homotopy equivalence; similarly, a fibration that is a homotopy equivalence can be endowed with a structure of a *strong* homotopy equivalence. The two theorems below will be used in the next chapter.

As usual, we assume that $\mathcal{I}$ contains $[\delta_0, \delta_1]$ and is closed under $\hat{\otimes}$.

Theorem 2.27 ("Vogt's lemma" for fibrations)**.** *If $p : E \to X$ is a fibration and a homotopy equivalence with E and X cofibrant and X fibrant, then p can be endowed with a structure of a strong homotopy equivalence.*

Proof. Suppose p is a homotopy equivalence with a homotopy inverse $s' : X \to E$ and homotopies $\varphi : s'p \sim \mathrm{id}_E$ and $\psi : ps' \sim \mathrm{id}_X$. First of all, we will replace s' with a homotopy inverse $s : X \to E$, which is a section, thus getting rid of a homotopy ψ. After that we replace a homotopy φ with a homotopy that will make p a strong homotopy equivalence.

Consider a lifting problem

$$\begin{array}{ccc} \{0\} \times X & \xrightarrow{s'} & E \\ {\scriptstyle \delta_0 \times X = \delta_0 \hat{\otimes} \bot_X}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ I \times X & \xrightarrow{\psi} & X \end{array}$$

We put $s = h \circ (\delta_1 \times X)$. Then h is a homotopy $h : s' \sim s$ and $h.p : s'p \sim sp$. By reversal and composition of homotopies we have a homotopy $sp \sim \mathrm{id}_E$. Furthermore, $p \circ s = \mathrm{id}_X$ and so we can obtain a trivial homotopy $\pi_2 : \mathrm{id}_X \sim ps$.

So we can assume that the homotopy inverse of p is a section of p and the homotopy ψ is trivial. Now we want to replace φ with a strong homotopy. This would mean that the new homotopy must satisfy

$$\begin{array}{ccc} I \times E & \xrightarrow{I \times p} & I \times X \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \epsilon \times X} \\ E & \xrightarrow[p]{} & X \end{array}$$

which is equivalent to χ being fibrewise homotopy equivalence $\chi : \mathrm{id}_E \sim_X sp$

$$\begin{array}{ccc} I \times E & \xrightarrow{\pi_2} & E \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle p} \\ E & \xrightarrow[p]{} & X \end{array}$$

that is, the homotopy χ is mapped to a trivial homotopy under p.

To obtain such a homotopy, consider an "open box" in E^E

$$\begin{array}{ccc} \mathrm{id}_E & \overset{\bar{\varphi}}{\rightsquigarrow} & sp = spsp \\ \| & & \uparrow{\scriptstyle \overline{sp.\varphi}} \\ \mathrm{id}_E & & sp \end{array}$$

which corresponds to a map $(I \times \{1\}) \cup (\{0\} \times I + \{1\} \times I) \xrightarrow{[\overline{\varphi}, \overline{\pi_2}, \overline{sp.\varphi}]} E^E$ (π_2 is a trivial homotopy $\mathrm{id}_{\mathrm{id}_E} : \mathrm{id}_E \rightsquigarrow \mathrm{id}_E$). Under the image of p^E (i.e. post-composition with p), this "open box" corresponds to an "open box" in X^E:

$$\begin{array}{ccc} p & \overset{\overline{p.\varphi}}{\rightsquigarrow} & psp = p \\ \| & & \uparrow{\scriptstyle \overline{psp.\varphi} = \overline{p.\varphi}} \\ p & & psp = p \end{array}$$

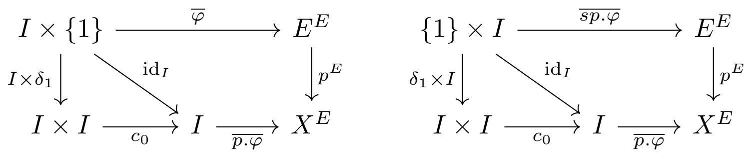
In the presence of connections, this box can be "closed"

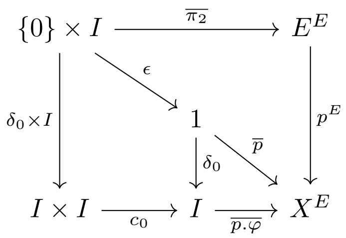
$$\begin{array}{ccc} p & \overset{\overline{p.\varphi}}{\rightsquigarrow} & p \\ \| & & \uparrow{\scriptstyle \overline{p.\varphi}} \\ p & = & p \end{array}$$

Formally, this square corresponds to a map $I \times I \xrightarrow{c_0} I \xrightarrow{\overline{p.\varphi}} X^E$. Thus, we obtain a commutative square

$$\begin{array}{ccc} (I \times \{1\}) \cup (\{0\} \times I + \{1\} \times I) & \xrightarrow{[\overline{\varphi}, \overline{\pi_2}, \overline{sp.\varphi}]} & E^E \\ {\scriptstyle [\delta_0, \delta_1] \hat{\otimes} \delta_1}\downarrow & & \downarrow{\scriptstyle p^E} \\ I \times I & \xrightarrow[c_0]{} I \xrightarrow[\overline{p.\varphi}]{} & X^E \end{array} \tag{2.3}$$

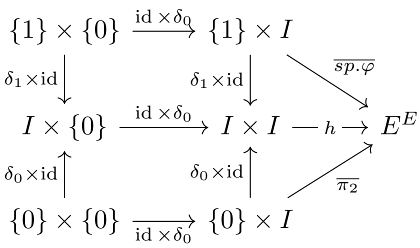
to verify the commutativity of the diagram above it suffices to check the commutativity "point-wise" like in the diagrams below





where the upper triangle in the lowest diagram commutes because $\overline{p^E\overline{\pi_2}} = p\pi_2 = p(\epsilon \times E) = \overline{\overline{p}}\epsilon$.

Finally, by remark 2.7, p^E is a fibration, so the diagram (2.3) has a filler $h : I \times I \to E^E$. Take $\chi = \overline{h \circ (I \times \delta_0)}$; then we can verify that χ is indeed a homotopy $\mathrm{id}_E \rightsquigarrow sp$ by transposing the following commutative diagram



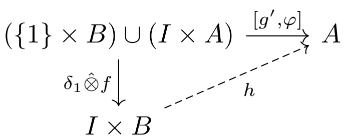
It is also straightforward to check that χ is mapped to the trivial homotopy under p.

□

Theorem 2.28 ("Vogt's lemma" for cofibrations)**.** *If $f : A \to B$ is a cofibration (that is, an element of $\mathcal{I}$) and a homotopy equivalence with A and B being fibrant and B cofibrant , then f can be endowed with a structure of a strong homotopy equivalence.*

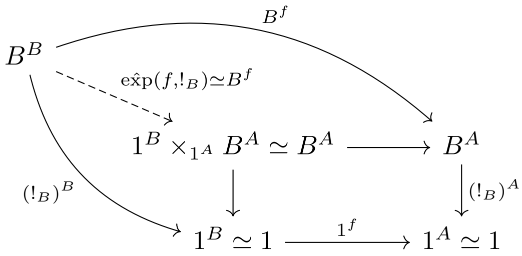
Proof. Suppose f is a homotopy equivalence with a homotopy inverse $g' : B \to A$ and homotopies $\varphi : \mathrm{id}_A \sim g'f$ and $\psi : \mathrm{id}_B \sim fg'$. Just like in the previous theorem, we provide the proof in two steps. First, we replace g' with a one-sided inverse g, making f into a section. Then, we replace the homotopy ψ in such a way that f will become a strong deformation section.

Since $\varphi \circ (\delta_1 \times A) = g' \circ f$, we have a well-defined map $[g', \varphi] : (\{1\} \times B) \cup (I \times A) \to A$. Since A is fibrant, the lifting problem below has a solution $h : I \times B \to A$

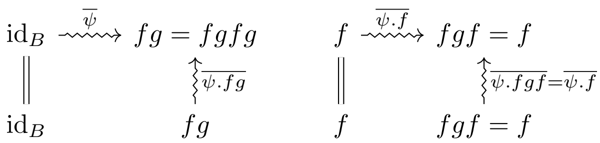


Then we define $g := h \circ (\delta_0 \times B)$. By construction, $h : g' \sim g$. We also have $gf = h(\delta_0 \times B)f = h(I \times f)(\delta_0 \times A) = \varphi(\delta_0 \times A) = \mathrm{id}_A$, so (f, g) is a section-retraction pair. Furthermore, $f.h = f \circ h : fg \sim fg'$ is a homotopy; by composition and reversal of homotopies (lemma 2.14) we have $\psi * f.h : fg \sim \mathrm{id}_B$, so g is also a homotopy inverse of f.

Therefore, we can assume that f is a homotopy equivalence with a homotopy inverse g, such that the homotopy $\varphi : \mathrm{id}_A \sim gf$ is trivial ($\varphi = \pi_2$). Let ψ be the second homotopy $\psi : \mathrm{id}_B \sim fg$. Now we will replace ψ with another homotopy χ, which will make f a strong homotopy equivalence. First of all we have to note the map $B^f : B^B \to B^A$ is isomorphic to the map $\hat{\exp}(f, !_B)$; this can be seen by examining the diagram of the $\hat{\exp}$ construction:



By proposition 2.6, $\hat{\exp}(f, !_B)$ is a fibration if f is a cofibration and $!_B$ is a fibration; those are exactly our assumptions. To sum up, $B^f : B^B \to B^A$ is a fibration. The rest of the proof goes similarly to the proof of theorem 2.28. Specifically, we form an "open box" in B^B (depicted on the left below); then we can look at that "open box" under the image of $(-).f = B^f$ in B^A (depicted on the right below).



The "open box" in B^A can be filled using connections. Thus, we can lift the filler to B^B along B^f. The lower part of the filled box will be a transpose of a homotopy $\chi : \mathrm{id}_B \sim fg$, such that $\chi.f = \mathrm{id}_f$ or, in other words, $\chi(I \times f) = f \circ \pi_2$, which is exactly the condition we want for f to be strong homotopy equivalence.

□

Notes

This chapter contained the basic blocks that we will use for building a model structure, and a model of type theory. Starting with a fairly simple assumption we introduced important $\mathcal{I}$-fibrations, homotopies, and homotopy equivalences.

The notion of an $\mathcal{I}$-fibration is defined with the help of a Leibniz adjunction, and is set up in such a way that $\mathcal{I}$-fibrations have an "open box" filling property, and are closed under exponents. Furthermore, a very natural notion of homotopy induced by the interval object behaves "as expected" on fibrant objects, just like the strengthened notion of a fibrewise homotopy – which is a generalization of standard topological notion of a homotopy rel endpoints. Specifically, the homotopy relation is a congruence on the hom-sets between cofibrant-fibrant objects.

From the notion of homotopy we defined, in a standard manner, homotopy equivalences and (strict) *strong* homotopy equivalences. As it turns out, a homotopy equivalence that is an $\mathcal{I}$-fibration or a cofibration can be endowed with a structure of a strong homotopy equivalence.

Since homotopy relation is a congruence, homotopy equivalences can be inverted. This is one of the axioms of a model category structure. In the next chapter we will show how exactly all those definitions fit together into a Quillen model category.

Chapter 3

Quillen Model Structure

The notion of a uniform $\mathcal{I}$-fibration by Gambino & Sattler is inspired by Cisinski's work [10] on model structures on Grothendieck topoi, in which the cofibrations are monomorphisms. However, as we want to build a model structure on the effective topos, we cannot use Cisinski's construction, because the effective topos is not cocomplete. We will, however, manage to obtain a model structure on a subcategory of $\mathcal{E}ff$ consisting of fibrant objects.

The aim of this chapter is to prove the following fact: given a topos $\mathcal{C}$, we can take the class $\mathcal{I}$ to be the class of all monomorphisms of $\mathcal{C}$. Then, there exists a model category structure on the subcategory of fibrant (in the sense of uniform $\mathcal{I}$-fibrations) objects $\mathcal{C}_f \hookrightarrow \mathcal{C}$, where cofibrations are exactly the monomorphisms, fibrations are uniform $\mathcal{I}$-fibrations, and weak equivalences are homotopy equivalences.

The crucial part of that model structure is the (acyclic cofibrations, fibrations) weak factorisation system. The existence of such a factorisation system does not depend on the fact that $\mathcal{C}$ is a topos. In fact, for a closed bi-Cartesian category $\mathcal{C}$, such a factorisation system exists on $\mathcal{C}_f$ for a class $\mathcal{I}$, satisfying the following restrictions:

1. Every map from $\mathcal{I}$ is a monomorphism, and $\mathcal{I}$ contains sections.
2. $\emptyset \to X \in \mathcal{I}$ for each $X \in \mathcal{C}$ (i.e., all objects are cofibrant)
3. $[\delta_0, \delta_1] : 1 + 1 \to I$ and $\delta_0, \delta_1 : 1 \to I$ are in $\mathcal{I}$.
4. $\mathcal{I}$ is closed under Leibniz product and under retracts.

The second weak-factorisation system, (cofibrations, acyclic fibrations) are obtained using the fact that $\mathcal{C}$ is a topos. On any topos, we can construct a factorisation system where the left maps are monomorphisms. We will show that acyclic fibrations are exactly trivial uniform $\mathcal{I}$-fibrations, and thus those maps that have the right lifting property against monomorphism.

The structure of this chapter is as follows. In the next section we prove that homotopy equivalences can be formaly inverted. In sections 3.2 and 3.3 we present $(\mathcal{W} \cap \mathcal{C}of, \mathcal{F}ib)$ and $(\mathcal{C}of, \mathcal{W} \cap \mathcal{F}ib)$ factorisation systems, respectively.

3.1 Weak homotopy equivalences

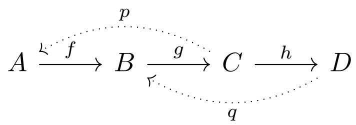
For class $\mathcal{W}$ of homotopy equivalences to satisfy the axioms of a model category we want:

1. Homotopy equivalences contain all isomorphisms and satisfy the 2-out-of-3 property;
2. $(\mathcal{C}of \cap \mathcal{W})^{\pitchfork} = \mathcal{F}ib$ and $(\mathcal{C}of \cap \mathcal{W}) = {}^{\pitchfork}\mathcal{F}ib$, and any map f can be factored as gh with $h \in (\mathcal{C}of \cap \mathcal{W})$ and $g \in \mathcal{F}ib$;
3. $\mathcal{C}of^{\pitchfork} = \mathcal{F}ib \cap \mathcal{W}$ and $\mathcal{C}of = {}^{\pitchfork}(\mathcal{F}ib \cap \mathcal{W})$, and any map f can be factored as gh with $h \in \mathcal{C}of$ and $g \in (\mathcal{F}ib \cap \mathcal{W})$;

First of all, note that all isomorphisms are trivially homotopy equivalences. Secondly, the 2-out-of-3 property is implied by the following proposition.

Proposition 3.1. *Homotopy equivalences satisfy the* two-out-of-six *property; that is, if $f : A \to B$, $g : B \to C$ and $h : C \to D$ are composable arrows between fibrant objects, and gf, hg are homotopy equivalences, then so are f, g, h, and hgf.*

Proof. This follows from the fact that $\sim$ is a congruence. Suppose $p : C \to A$ is a homotopy inverse of gf, and $q : D \to B$ is a homotopy inverse of hg.



Then we claim that pgq is a homotopy inverse of hgf, pg is a homotopy inverse of f, fp is a homotopy inverse of g, and gq is a homotopy inverse of h.

We employ a form of equational reasoning, employing proposition 2.15.

1. hgf is a homotopy equivalence:
   (a) $hgf(pgq) = h(gfp)q \sim hgq = (hg)q \sim \mathrm{id}_D$
   (b) $(pgq)hgf = pg(qhg)f \sim pgf = p(gf) \sim \mathrm{id}_A$
2. f is a homotopy equivalence:
   (a) $f(pg) \sim (qhg)fpg = qh(gfp)g \sim qhg \sim id_B$
   (b) $(pg)f = p(gf) \sim \mathrm{id}_A$
3. g is a homotopy equivalence:
   (a) $g(fp) = (gf)p \sim \mathrm{id}_C$
   (b) $(fp)g \sim (qhg)fpg \sim qhgfpg(qhg) = q(hgfpgq)hg \sim qhg \sim \mathrm{id}_B$
4. h is a homotopy equivalence:
   (a) $h(gq) = (hg)q \sim \mathrm{id}_D$
   (b) $(gq)h \sim gqh(gfp) = g(qhg)fp \sim gfp \sim \mathrm{id}_C$

□

Since we are working in the category $\mathcal{C}_f$ of fibrant object, we can make use of theorems in section 2.2.4, which states that a homotopy equivalence that is an $\mathcal{I}$-fibration or a cofibration can be replaced by a strong homotopy rquivalence. Writing $\mathcal{S}$ for a class of strong homotopy equivalences, that means that $\mathcal{S} \cap \mathcal{F}ib = \mathcal{W} \cap \mathcal{F}ib$ and $\mathcal{S} \cap \mathcal{C}of = \mathcal{W} \cap \mathcal{C}of$; thus, in the rest of the chapter we can freely assume that acyclic fibrations and acyclic cofibrations are strong homotopy equivalences.

3.2 (Acyclic Cofibration, Fibration) factorisation system

In this section we establish and study the (acyclic cofibrations, fibrations) factorisation system.

That involves showing that $(\mathcal{W} \cap \mathcal{C}of)^{\pitchfork} = \mathcal{F}ib$, $(\mathcal{W} \cap \mathcal{C}of) = {}^{\pitchfork}\mathcal{F}ib$, and presenting a factorisation of a map as an acyclic cofibration followed by a fibration.

To show that $(\mathcal{W} \cap \mathcal{C}of)^{\pitchfork} = \mathcal{F}ib$ holds, we will first establish that $\mathcal{F}ib \subseteq (\mathcal{I} \cap \mathcal{W})^{\pitchfork}$, that is, that every uniform $\mathcal{I}$-fibration $f \in \mathcal{I}\text{-}\mathcal{F}ib$ has the right lifting property against maps from $\mathcal{I}$ that are homotopy equivalences (theorem 3.2). For the other direction

$\mathcal{F}ib \supseteq (\mathcal{I} \cap \mathcal{W})^{\pitchfork}$ we will employ the so called "retract argument" (proposition 3.5); but first we will need to describe the functorial factorisation.

After establishing the first equation, we can rewrite the second equation $(\mathcal{I} \cap \mathcal{W}) = {}^{\pitchfork}\mathcal{F}ib$ as $(\mathcal{I} \cap \mathcal{W}) = {}^{\pitchfork}((\mathcal{I} \cap \mathcal{W})^{\pitchfork})$. The inclusion $(\mathcal{I} \cap \mathcal{W}) \subseteq {}^{\pitchfork}((\mathcal{I} \cap \mathcal{W})^{\pitchfork})$ always holds, as $(-)^{\pitchfork} \dashv {}^{\pitchfork}(-)$ form an adjunction. For the other direction, which is described in proposition 3.6, we employ the retract argument once again.

3.2.1 Acyclic cofibrations and orthogonality

First of all, we establish the inclusion $\mathcal{F}ib \subseteq (\mathcal{W} \cap \mathcal{I})^{\pitchfork}$. Per discussion in section 2.2.4, it suffices to show the following.

Theorem 3.2. *If a map f is a $\mathcal{I}$-$\mathcal{F}ib$, then f has the right lifting property against cofibrations that are strong homotopy equivalences.*

Proof. Let $u_i \in \mathcal{I}$ be a strong homotopy equivalence, and suppose there is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ {\scriptstyle u_i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow{k} & Y \end{array}$$

Since u_i is a strong homotopy equivalence, there is a map $g : B \to A$ and homotopies

$$\begin{array}{ccccc} A & \xrightarrow{\delta_0 \times A} & I \times A & \xleftarrow{\delta_1 \times A} & A \\ {\scriptstyle u_i}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} & \diagup\!\!\diagup & \\ B & \xrightarrow[g]{} & A & & \end{array} \qquad \begin{array}{ccccc} B & \xrightarrow{\delta_0 \times B} & I \times B & \xleftarrow{\delta_1 \times B} & B \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \psi} & \diagup\!\!\diagup & \\ A & \xrightarrow[u_i]{} & B & & \end{array}$$

Furthermore, the homotopy equivalence is strong:

$$\begin{array}{ccc} I \times A & \xrightarrow{I \times u_i} & I \times B \\ {\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ A & \xrightarrow{u_i} & B \end{array}$$

First, we construct a commutative square

$$\begin{array}{ccc} (\{0\} \times B) \cup (I \times A) & \xrightarrow{h \circ [g,\varphi]} & X \\ {\scriptstyle \delta_0 \hat{\otimes} u_i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ I \times B & \xrightarrow{k \circ \psi} & Y \end{array}$$

One can verify that $[g, \varphi] : (\{0\} \times B) \cup (I \times A)$ is a well-defined map; it is possible to check that the diagram commutes "component-wise":

$$f \circ h \circ g = k \circ u_i \circ g = k \circ (\psi \circ (\delta_0 \times B))$$

$$f \circ h \circ \varphi = k \circ u_i \circ \varphi = k \circ (\psi \circ (I \times u_i))$$

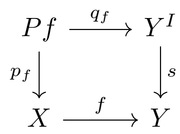
where the last equality holds because the homotopy equivalence for u_i is strong. Thus the commutative diagram above has a filler $m : I \times B \to X$. We claim that $m \circ (\delta_1 \times B)$ is a filler for the original diagram. For this it suffices to check that $f \circ m \circ (\delta_1 \times B) = k \circ \psi \circ (\delta_1 \times B) = k$ and $m \circ (\delta_1 \times B) \circ u_i = m \circ (I \times u_i) \circ (\delta_1 \times A) = h \circ \varphi \circ (\delta_1 \times A) = h$.

□

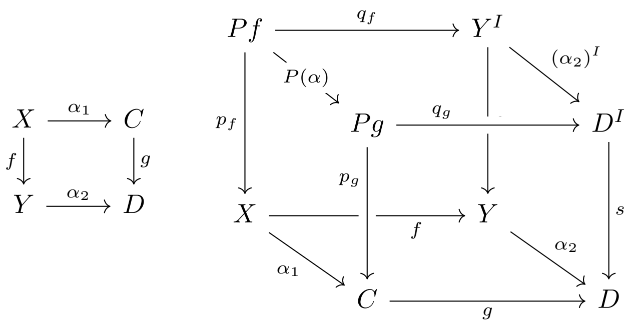
In order to verify the other inclusion $\mathcal{F}ib \supseteq (\mathcal{W} \cap \mathcal{I})^{\pitchfork}$ we will employ the retract argument, by factoring a map in $(\mathcal{W} \cap \mathcal{I})^{\pitchfork}$ as an acyclic cofibration followed by an $\mathcal{I}$-fibration. However, before we do that we must present the functorial factorisation.

3.2.2 Functorial factorisation

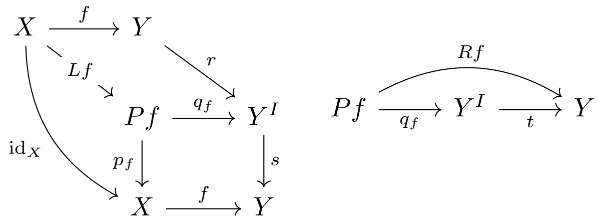
The factorisation is given by a functor $P : \mathcal{C}^2 \to \mathcal{C}$ that assigns to each $f : X \to Y$ a "mapping cocylinder" Pf obtained via a pullback



for a commutative square $\alpha = (\alpha_1, \alpha_2) : f \to g$, the arrow $P(\alpha) : Pf \to Pg$ is defined by the universal property of the pullback:



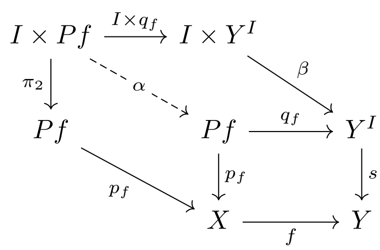
We factor a map $f : X \to Y$ factors as $Lf \circ Rf$. The left map $Lf = \langle \mathrm{id}_X, r \circ f \rangle$ is obtained from the universal property of the pullback, and the right map Rf is defined to be the composite $t \circ q_f$:



Intuitively, Pf is an object of Y-paths starting at $f(x)$, $\Sigma_{x:X,y:Y} \mathrm{Id}(f(x), y)$. Then Lf sends every x to a constant path at $f(x)$. The map Rf just returns the endpoint of a path, corresponding to the projection $(x, y, p) : \Sigma_{x:X,y:Y} \mathrm{Id}(f(x), y) \mapsto y : Y$.

Proposition 3.3. *Lf is a monomorphism and a homotopy equivalence*

Proof. Lf has a retraction p_{Rf}, which makes it a monomorphism; furthermore, p_{Rf} is also its homotopy inverse. To show this we need a homotopy $\alpha : \mathrm{id}_{Pf} \sim Lf \circ p_f$, which can be intuitively seen as a way of contracting a path $f(x) \rightsquigarrow y$ to a constant path $f(x) \rightsquigarrow f(x)$. We obtain α from the pullback



where β is a homotopy $\beta : \mathrm{id}_{X^I} \sim r \circ s$ defined in remark 2.25.

To verify that the homotopy has the required source and target, it suffices to check the composites $\{0\} \times Pf \rightarrowtail I \times Pf \xrightarrow{\alpha} Pf$ and $\{1\} \times Pf \rightarrowtail I \times Pf \xrightarrow{\alpha} Pf$ up to post-composition with q_f and p_f.

In case of $\{0\} \times Pf \rightarrowtail I \times Pf$ we have

- $p_f \circ \alpha \circ (\delta_0 \times Pf) = p_f \circ \pi_2 \circ (\delta_0 \times Pf) = p_f = p_f \circ \mathrm{id}_{Pf}$
- $q_f \circ \alpha \circ (\delta_0 \times Pf) = \beta \circ (I \times q_f) \circ (\delta_0 \circ \mathrm{id}) = \beta \circ (\delta_0 \times \mathrm{id}) \circ q_f = \mathrm{id}_{Y^I} \circ q_f = q_f \circ \mathrm{id}_{Pf}$

In case of $\{1\} \times Pf \rightarrowtail I \times Pf$ we have

- $p_f \circ \alpha \circ (\delta_1 \times Pf) = p_f = p_f \circ Lf \circ p_f$
- $q_f \circ \alpha \circ (\delta_1 \times Pf) = \beta \circ (\delta_1 \times \mathrm{id}) \circ q_f = r \circ s \circ q_f = r \circ f \circ p_f = q_f \circ Lf \circ p_f$

□

Proposition 3.4. *Rf is a fibration*

Proof. The object Pf can be equivalently described by a pullback of $\langle s, t\rangle : Y^I \to Y$ along $f \times \mathrm{id}_Y : X \times Y \to Y \times Y$. To see this, note that the original description of Pf fits into the diagram below

$$\begin{array}{ccc} P'f & \xrightarrow{\Gamma_2} & Y^I \\ {\scriptstyle \Gamma_1}\downarrow & & \downarrow{\scriptstyle \langle s,t\rangle} \\ X \times Y & \xrightarrow{f \times \mathrm{id}_Y} & Y \times Y \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ X & \xrightarrow{f} & Y \end{array}$$

where both of the inner squares are pullbacks; hence the outer square is a pullback and $Pf \simeq P'f$.

Then, $Rf = t \circ q_f$ can be equivalently described as $\pi_2 \circ \langle s, t\rangle \circ \Gamma_2 = \pi_2 \circ (f \times \mathrm{id}_Y) \circ \Gamma_1 = \pi_2 \circ \Gamma_1$. By propositions 2.5 and 2.8, Γ_1 is a fibration. In fact, it represents a type $x : X, y : Y \vdash \mathrm{Id}_Y(f(x), y)$. By remark 2.11, $\pi_2 : X \times Y \to Y$ is a fibration; thus the composite $Rf = \pi_2 \circ \Gamma_1$ is a fibration. □

Now that we have established the factorization, we want to verify that $\mathcal{F}ib \supseteq (\mathcal{W} \cap \mathcal{C}of)^{\pitchfork}$. For that we make use of the factorisation discussed in this subsection and the standard "retract argument" (see, e.g. [18, Proposition 7.2.2]).

Proposition 3.5. *If a map f has the right lifting property against acyclic cofibrations, then f is an $\mathcal{I}$-fibration.*

Proof. Given a map $f \in \mathcal{W} \cap \mathcal{C}of^{\pitchfork}$, we can factor f as $Lf \circ Rf$, where $Rf \in \mathcal{F}ib$ and $Lf \in \mathcal{W} \cap \mathcal{C}of$; then, the diagram below on the right has a diagonal filler

$$\begin{array}{ccccc} X & & \xrightarrow{f} & & Y \\ & {\scriptstyle Lf}\searrow & & \nearrow{\scriptstyle Rf} & \\ & & Pf & & \end{array} \qquad \begin{array}{ccc} X & = & X \\ {\scriptstyle Lf}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle f} \\ Pf & \xrightarrow[Rf]{} & Y \end{array}$$

One can then easily check that the following diagram commutes, and therefore f is a retract of Rf.

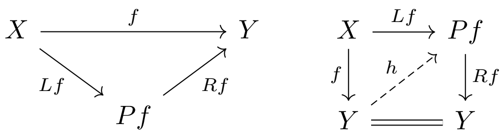
$$\begin{array}{ccccc} X & \xrightarrow{Rf} & Pf & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} & & \downarrow{\scriptstyle f} \\ Y & = & Y & = & Y \end{array}$$

Since fibrations are closed under retracts (as any class defined by a lifting property), we can conclude that $f \in \mathcal{F}ib$. □

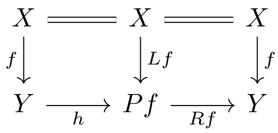
Thus, we have $\mathcal{F}ib = (\mathcal{C}of \cap \mathcal{W})^{\pitchfork}$. It remains to establish that ${}^{\pitchfork}\mathcal{F}ib = (\mathcal{C}of \cap \mathcal{W})$. As we have already discussed, this boils down to showing the inclusion ${}^{\pitchfork}\mathcal{F}ib \subseteq \mathcal{C}of \cap \mathcal{W}$. For this we empoloy the retract argument once again.

Proposition 3.6. *If f has the left lifting property against $\mathcal{I}$-fibrations, then f is an acyclic cofibration.*

Proof. Given a map $f \in {}^{\pitchfork}\mathcal{F}ib$, we can factor f as $Lf \circ Rf$, where $Rf \in \mathcal{F}ib$; then, the diagram below on the right has a diagonal filler



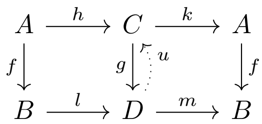
Similarly to the proof of the previous proposition, this implies that f is a retract of Lf:



It is clear that $\mathcal{C}of$ is closed under the retracts; to conclude that $f \in \mathcal{C}of \cap \mathcal{W}$ it is sufficient to establish that $\mathcal{W}$ is closed under retracts, which we will prove in the next proposition (Proposition 3.7). ☐

Proposition 3.7. *Homotopy equivalences are preserved under retracts.*

Proof. Let g be a homotopy equivalence with a homotopy inverse u, and consider the following retract diagram:



Then $(k \circ u \circ l)$ is a homotopy inverse of f:

$$k \circ u \circ l \circ f = k \circ u \circ g \circ h \sim k \circ h = \mathrm{id}_A$$

$$f \circ k \circ u \circ l = m \circ g \circ u \circ l \sim m \circ l = \mathrm{id}_B$$

☐

This sums up the description of the (acyclic cofibrations, fibrations) factorisation system.

3.3 (Mono, Acyclic Fibration) factorisation system

The second factorisation system is a $(Mono, Mono^{\pitchfork})$ factorisation system, as it is present in any topos via a *partial map classifier* (see [7, Section 4.4] for a detailed description in the context of algebraic weak factorisation systems). After describing this factorization system, we will have to show that $(\mathcal{C}of, \mathcal{F}ib \cap \mathcal{W}) = (Mono, Mono^{\pitchfork})$. Since in our case, $Mono^{\pitchfork} = Triv\mathcal{F}ib$, we will show that trivial fibrations are exactly those fibrations that are homotopy equivalences.

To explain the $(Mono, Mono^{\pitchfork})$ factorisation system, consider a terminal map $X \to 1$ that we want to factor. The factorisation would consist of a monomorphism into an injective object. We can show that the partial map classifier $\tilde{X}$, which classifies partial maps into X, is a good choice. First of all, $\tilde{X}$ comes with a "singleton" monomorphism $\eta : X \to \tilde{X}$, and the composite $X \rightarrowtail \tilde{X} \to 1$ is clearly equal to the terminal map $X \to 1$. It now remains to show that $\tilde{X}$ has the right lifting property against monomorphisms (that is, $\tilde{X}$ is injective). Suppose we have a diagram as below, for which we have to find a filler.

$$\begin{array}{ccc} A & \xrightarrow{f} & \tilde{X} \\ {\scriptstyle i}\big\downarrow & & \\ B & & \end{array}$$

The map $f : A \to \tilde{X}$ represents some partial map f', which is a pullback of η along f.

$$\begin{array}{ccc} A' & \xrightarrow{f'} & X \\ {\scriptstyle i'}\big\downarrow & & \big\downarrow{\scriptstyle \eta} \\ A & \xrightarrow{f} & \tilde{X} \end{array}$$

One can view partial map $\begin{array}{ccc} A' & \xrightarrow{f'} & X \\ {\scriptstyle i'}\big\downarrow & & \\ A & & \end{array}$ defined on the domain $A' \rightarrowtail A$ as a partial map defined on the domain $A' \rightarrowtail B$: $\begin{array}{ccc} A' & \xrightarrow{f'} & X \\ {\scriptstyle i \circ i'}\big\downarrow & & \\ B & & \end{array}$. Thus, we have an induced map $g : B \to \tilde{X}$ making the diagram on the left below a pullback. However, as one can verify with a help of a diagram chase, g also makes the diagram on the right below a pullback.

$$\begin{array}{ccc} A' & \xrightarrow{f'} & X \\ {\scriptstyle i'}\big\downarrow & & \\ A & & \big\downarrow{\scriptstyle \eta} \\ {\scriptstyle i}\big\downarrow & & \\ B & \xrightarrow{g} & \tilde{X} \end{array} \qquad \begin{array}{ccccc} A' & & \xrightarrow{f'} & & X \\ {\scriptstyle i'}\big\downarrow & & & & \big\downarrow{\scriptstyle \eta} \\ A & \underset{i}{\rightarrowtail} & B & \xrightarrow{g} & \tilde{X} \end{array}$$

This means that both $g \circ i$ and f represent the same partial function $B \rightharpoonup X$, which means that they are equal. Hence g is the required filler for the original lifting problem. Since each slice of a topos is again a topos, and every topos has a partial map classifier (see, e.g. [24, Theorem 1.26]), we can factor a map $X \to Y$ by repeating the same argument in the slice category over Y.

Thus, we have a factorisation of any map in a topos by a monomorphism followed by a map that has the right lifting property against monomorphisms. Now we are to show that $Mono^{\pitchfork} = Triv\mathcal{F}ib = \mathcal{F}ib \cap \mathcal{W}$. Once again, we make use of the fact that $\mathcal{F}ib \cap W = \mathcal{F}ib \cap \mathcal{S}$.

Theorem 3.8. *If $p : E \to X$ is a uniform fibration and a strong homotopy equivalence, then p is a trivial uniform fibration.*

Proof. The proof is similar to [10, Lemma 2.25], except that we don't require p to be a strong deformation retraction. Let

$$\begin{array}{ccc} A & \xrightarrow{k} & E \\ {\scriptstyle u_i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{h} & X \end{array} \tag{3.1}$$

be a commutative square for $i \in \mathcal{I}$. We are going to construct a lifting $B \to E$. Since p is a strong homotopy equivalence, it has a homotopy inverse $q : X \to E$ and homotopies

φ and ψ:

$$\begin{array}{ccccc} E & \xrightarrow{\delta_0\times E} & I\times E & \xleftarrow{\delta_1\times E} & E \\ & \searrow^{=} & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle p} \\ & & E & \xleftarrow[q]{} & X \end{array} \qquad \begin{array}{ccccc} X & \xrightarrow{\delta_0\times X} & I\times X & \xleftarrow{\delta_1\times X} & X \\ & \searrow^{=} & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle q} \\ & & X & \xleftarrow[p]{} & E \end{array} \tag{3.2}$$

Then consider the following diagram

$$\begin{array}{ccccccccc} \{0\}\times A & \xrightarrow{\delta_0\times A} & I\times A & \rightarrowtail & (\{1\}\times B)\cup(I\times A) & & \xrightarrow{[q\circ h,\varphi\circ(I\times k)]} & & E \\ \downarrow{\scriptstyle \{0\}\times u_i} & & \downarrow{\scriptstyle I\times u_i} & & \downarrow{\scriptstyle \delta_1\hat{\otimes}u_i} & & & & \downarrow{\scriptstyle p} \\ \{0\}\times B & \xrightarrow[\delta_0\times B]{} & I\times B & = & I\times B & \xrightarrow[I\times h]{} & I\times X & \xrightarrow[\psi]{} & X \end{array} \tag{3.3}$$

The bottom composite $\{0\}\times B \simeq B \to X$ of the chain of morphisms is h, by the commutativity of the right-hand diagram (3.2). The composite $\{0\}\times A \simeq A \to E$ of the chain of morphisms at the top of the diagram is $\varphi\circ(I\times k)\circ(\delta_0\times A) = k$ by the commutativity of the left-hand diagram (3.2). Thus, if all the inner squares commute, then the filler $I\times B \to E$ of the rightmost square will induce a filler for (3.1) by the precomposition with $\delta_0\times B$.

The leftmost inner square commutes by the naturality of $\delta_0\times(-)$, the middle inner square commutes by the definition of Leibniz product. It remains to show that

1. $[q\circ h, \varphi\circ(I\times k)] : (\{1\}\times B)\cup(I\times A)\to E$ is a well-defined map
2. the rightmost inner square commutes

To prove the first item, we must verify that

$$\{1\}\times A \xrightarrow{\{1\}\times u_i} \{1\}\times B \simeq B \xrightarrow{h} X \xrightarrow{q} E$$

is equal to

$$\{1\}\times A \xrightarrow{\delta_1\times A} I\times A \xrightarrow{I\times k} I\times E \xrightarrow{\varphi} E$$

For this it suffices to note that the following diagram commutes

$$\begin{array}{ccccc} & \overset{u_i}{\nearrow} & B & \overset{h}{\searrow} & \\ \{1\}\times A\simeq A & \xrightarrow{k} & \{1\}\times E & \xrightarrow{p} & X \\ \downarrow{\scriptstyle \delta_1\times A} & & \downarrow{\scriptstyle \delta_1\times E} & & \downarrow{\scriptstyle q} \\ I\times A & \xrightarrow{I\times k} & I\times E & \xrightarrow{\varphi} & E \end{array}$$

The upper triangle is exactly diagram (3.1), the leftmost square commutes by the naturality of $\delta_1\times(-)$, and the rightmost square commutes by the definition of φ.

To prove the second item, it suffices to check the commutativity of the rightmost square in (3.3) up to precomposition with $\{1\}\times B \hookrightarrow (\{1\}\times B)\cup(I\times A)$ and with $I\times A \hookrightarrow (\{1\}\times B)\cup(I\times A)$, which can be witnessed by the commutative diagrams below.

$$\begin{array}{ccccc} \{1\}\times B & \xrightarrow{h} & \{1\}\times X & \xrightarrow{q} & E \\ \downarrow{\scriptstyle \delta_1\times B} & & \downarrow{\scriptstyle \delta_1\times X} & & \downarrow{\scriptstyle p} \\ I\times B & \xrightarrow[I\times h]{} & I\times X & \xrightarrow[\psi]{} & X \end{array} \qquad \begin{array}{ccccc} I\times A & \xrightarrow{I\times k} & I\times E & \xrightarrow{\varphi} & E \\ \downarrow{\scriptstyle I\times u_i} & & \downarrow{\scriptstyle I\times p} & & \downarrow{\scriptstyle p} \\ I\times B & \xrightarrow[I\times h]{} & I\times X & \xrightarrow[\psi]{} & X \end{array}$$

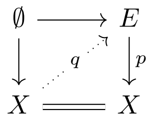
The rightmost square of the right-hand diagram is precisely the requirement that p is a *strong* homotopy equivalence.

□

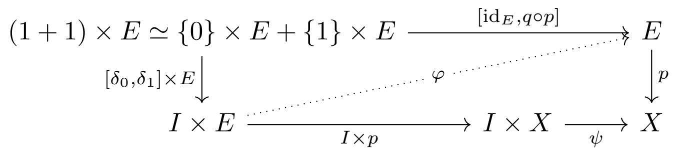
The converse holds as well.

Theorem 3.9. *If $p : E \to X$ is a trivial uniform fibration, then it is a strong homotopy equivalence.*

Proof. This is [10, Proposition 2.16]. First of all, p is a retraction and the section $X \to E$ is given by the filler



So $\psi = \pi : I \times X \to X$ is a homotopy $\mathrm{id}_X \sim p \circ q$. A homotopy $\varphi : \mathrm{id}_E \sim q \circ p$ is obtained from following lifting problem:



□

We can summarize the work of this chapter in the following theorem.

Theorem 3.10. *Let $\mathcal{C}$ be an elementary topos with an interval object I. We take $\mathcal{I}$ to be the class of all monomorphisms of $\mathcal{C}$ and we say that an object $X \in \mathcal{C}$ is fibrant if the unique map $X \to 1$ is an $\mathcal{I}$-fibration. There is a model category structure on the full subcategory $\mathcal{C}_f$ of fibrant objects of $\mathcal{C}$ in which*

- *Cofibrations are monomorphisms;*
- *Weak equivalences are homotopy equivalences w.r.t. I;*
- *Fibrations are $\mathcal{I}$-fibrations.*

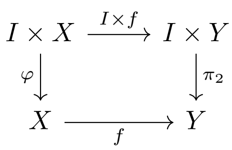
3.4 Application: weak groupoid structure

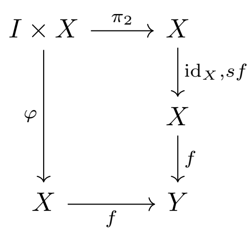
As an application of the QMS on $\mathcal{C}_f$, we will derive the weak groupoid structure on every object $X \in \mathcal{C}_f$. This is a general fact about fibrant objects in a model category where every object is cofibrant. See, e.g. [4, Appendix 1] or [13, Section 1.2.2]. The most important consequence of this groupoid structure is that the paths in a fibrant object X can be composed; this will be used in chapter 5.

For this we will use some results about *lower fillers*, similar to the propositions that hold in path categories [5, Section 2.6]. But first, we will need the following observation.

Proposition 3.11. *If $f : X \to Y$ is a fibration and a strong homotopy equivalence with a section s, then $s \circ f \sim_Y \mathrm{id}_X$.*

Proof. This is just a matter of unfolding the definitions. Because s is a section of f, we have $f \circ \mathrm{id}_X = f \circ (s \circ f)$. Let φ be the homotopy $s \circ f \sim \mathrm{id}_X$. Then, the diagram below on the left is the diagram that we obtain by the definition of a strong homotopy equivalence. It is equivalent to the diagram below on the right, which is the requirement for a fibrewise homotopy over Y.





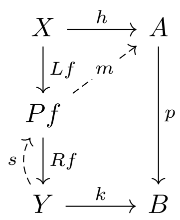
□

Lower fillers. Given a commutative square $\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}$, a *lower filler* is a map $h : C \to B$, such that $f \circ h = g$.

Proposition 3.12. *Let* $\begin{array}{ccc} X & \xrightarrow{h} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{k} & B \end{array}$ *be a commutative diagram, where p is a fibration, and f is a homotopy equivalence between fibrant objects. Then there is a lower filler $g : Y \to A$, such that $g \circ f \sim_B h$.*

Proof. Since X and Y are fibrant, we can apply the results of the previous section and factor f as a trivial cofibration Lf (monomorphism and a strong homotopy equivalence) followed by a fibration Rf. By the 2-out-of-3 property, Rf is a homotopy equivalence. By theorem 2.27, Rf has a section s which is also a (strong) homotopy inverse of Rf. By proposition 3.11, we have $s \circ Rf \sim_Y \mathrm{id}_{Pf}$.

Since Lf is a trivial cofibration, we have a diagonal filler $m : Pf \to A$ making the diagram below commute.



We claim that $m \circ s$ is the desired lower filler. Clearly, $p \circ m \circ s = k \circ Rf \circ s = k$. In addition, $p \circ m \circ s \circ f = k \circ Rf \circ s \circ f = k \circ f = p \circ h$, so we can apply proposition 2.20:

$$s \circ Rf \sim_Y \mathrm{id}_{Pf} \implies s \circ Rf \circ Lf = s \circ f \sim_Y Lf \implies m \circ s \circ f \sim_B m \circ Lf = h$$

$\square$

Weak groupoid structure. Given a fibrant object X, we have a "weak" groupoid structure on X where the maps are paths $I \to X$ (thus, the object of maps is X^I with projections $s, t : X^I \to X$), but some of the groupoid laws hold only up to homotopy over $X^I \xrightarrow{\langle s,t \rangle} X \times X$. We can derive a composition operation $c : X^I \times_X X^I \to X^I$, where $X^I \times_X X^I$ is the object of "composable paths", a pullback

$$\begin{array}{ccc} X^I \times_X X^I & \longrightarrow & X^I \\ \downarrow & & \downarrow{\scriptstyle s} \\ X^I & \xrightarrow{t} & X \end{array}$$

The composition operation should satisfy the groupoid axioms, with regard to the identity maps, given by $X \xrightarrow{X^\epsilon} X^I$. Specifically, we want

$$\begin{cases} c\langle rs, \mathrm{id}\rangle \sim_{X \times X} \mathrm{id} \\ c\langle \mathrm{id}, rt\rangle \sim_{X \times X} \mathrm{id} \\ c\langle c, \mathrm{id}\rangle \sim_{X \times X} c\langle \mathrm{id}, c\rangle \end{cases}$$

Proposition 3.13. *Given a fibrant object X, there is a composition operator $c : X^I \times_{X \times X} X^I \to X^I$ making X a groupoid with r as identity.*

Proof. The proof is basically that of [13, Proposition 1.12], but in a different categorical setting. Consider the following commutative square, where $\langle r, r\rangle$ is a homotopy equivalence and $\langle s, t\rangle$ is a fibration (by proposition 2.5).

$$\begin{array}{ccc} X & \xrightarrow{\quad r \quad} & X^I \\ {\scriptstyle \langle r,r\rangle}\downarrow & & \downarrow{\scriptstyle \langle s,t\rangle} \\ X^I \times_X X^I & \xrightarrow[\langle s\pi_1, t\pi_2\rangle]{} & X \times X \end{array}$$

By proposition 3.12 there is a lower filler c for this diagram, which is the composition operator.

One can show that $c \circ \langle rs, \mathrm{id}_{X^I}\rangle \sim_{X\times X} \mathrm{id}_{X^I} \sim_{X\times X} c \circ \langle \mathrm{id}_{X^I}, rt\rangle$ in the following way:

$$c \circ \langle rs, \mathrm{id}\rangle \circ r = c \circ \langle r, r\rangle \sim_{X\times X} r$$

By proposition 2.20, we can cancel out r (as it is a homotopy equivalence) and get $c \circ \langle rs, \mathrm{id}_{X^I}\rangle \sim_{X\times X} \mathrm{id}_{X^I}$. The other case is covered similarly. In the same manner we can show that c is associative (up to homotopy over $X \times X$).

□

Notes

In this chapter we have seen how to utilize the abstract framework from chapter 2 to obtain a model category structure on subcategory $\mathcal{C}_f$ of fibrant objects of a topos $\mathcal{C}$, in which cofibrations are monomorphisms and weak equivalences are homotopy equivalences in the sense of definition 2.21. The (cofibrations, trivial fibrations) factorisation (that is, $(\mathit{Mono}, \mathit{Mono}^{\pitchfork})$ factorisation) is obtained from the topos structure; specifically, from the partial map classifier. The other factorisation requires less structure on $\mathcal{C}$ and we employ a mapping cocylinder to construct it.

However, it is not enough to construct the factorisations, and we also showed that the three classes of maps "fit well" together. Specifically, we demonstrated that maps that have a lifting property against monomorphisms – that is, trivial fibrations – are exactly fibrations that are homotopy equivalences; and that the maps that have the left lifting property against fibrations are exactly monomorphisms that are homotopy equivalences.

Now we move onto the second part of the thesis, in which we apply the results of this part to the effective topos $\mathcal{E}ff$.

Part II

Constructing a model of type theory in the effective topos

Chapter 4

Effective topos

In this chapter we recall the definition and properties of the *effective topos* [22]. The effective topos $\mathcal{E}ff$ is an elementary topos (and thus provides a model for higher-order logic), which is not a Grothendieck topos. The effective topos is a "recursive universe" and the internal logic of $\mathcal{E}ff$ corresponds to higher-order realizability logic.

Even though we give a direct description of the effective topos in this chapter, it is worth noting that there are two important ways of constructing the effective topos from simpler categories. The first way to obtain $\mathcal{E}ff$ is through the tripos-to-topos construction [23]. A *tripos* is a hyperdoctrine with a "weak" powerobject. From a tripos we can construct a topos in which objects are the objects of the base category together with internal (w.r.t. the hyperdoctrine) equivalence relations, and the maps are internal functional relations that respect the equivalence relations. See [23] and [32] for the details on the construction.

The second way to obtain $\mathcal{E}ff$ is through a universal construction known as ex/reg completion [8]. For this, we start with a category Asm of assemblies (which we will identify as a subcategory of $\mathcal{E}ff$ in the last section of this chapter). The category of assemblies is sufficient for interpreting first-order intuitionistic logic, and higher-order logic, to some extent; it is, however, not a topos. To obtain the effective topos we must freely adjoin the quotients of equivalence relations in Asm.

While the constructions mentioned above are probably of interest to a category theorist, for our purposes it suffices to use an explicit definition.

This chapter is organized as follows. In the next section we recall the definition of number realizability and its proof-theoretic properties. In section 4.2 we establish a useful notation, which we dub as "realizability notation" or "realizability logic notation". This realizability logic is not the internal logic of $\mathcal{E}ff$, but rather is the internal logic of the effective tripos (as in [23]). The next two sections are devoted to the definition of the effective topos and some standard category-theoretic constructions in $\mathcal{E}ff$. In sections 4.5 and 4.6 we present the topos-theoretic constructions in $\mathcal{E}ff$ necessary for interpreting topos logic in $\mathcal{E}ff$, and we show how to describe those constructions in the realizability notation. In the final section we describe important classes of objects in $\mathcal{E}ff$: assemblies, modest sets, and uniform objects.

The goal of this chapter is to give an overview of the effective topos, and there are no new contributions in this chapter; the material presented here can be found in, e.g. [29, 38].

4.1 Kleene's number realizability

The proof-theoretic notion of *realizability* concerning first-order arithmetic first appeared in Kleene's seminal work [26]. This original Kleene's interpretation – which we shall call *Kleene realizability* or *number realizability* – can be seen as a way of extracting computational content from a constructive proof. For instance, from a proof of the statement

$\exists x.Q(x)$ (here, $Q(x)$ is a formula of arithmetic) one would want to extract a witness for the existential quantifier, a number n, s.t. $Q(n)$ holds. A *realizer* for a formula provides us with means of obtaining that information.

The fundamental notion and the main definition here is the number realizability relation $\underline{\mathsf{nr}}$, which links together numbers and formulas of Heyting arithmetic. This relation is defined by induction on the complexity of the formula.

- $n \ \underline{\mathsf{nr}}\ (p = q)$ if $\mathbf{HA} \vdash p = q$;
- $n \ \underline{\mathsf{nr}}\ (A \wedge B)$ if $\mathsf{p}_1 n \ \underline{\mathsf{nr}}\ A$ and $\mathsf{p}_2 n \ \underline{\mathsf{nr}}\ B$;
- $n \ \underline{\mathsf{nr}}\ (A \vee B)$ if $\mathsf{p}_1 n = 0$ and $\mathsf{p}_2 n \ \underline{\mathsf{nr}}\ A$ or $\mathsf{p}_1 n = 1$ and $\mathsf{p}_2 n \ \underline{\mathsf{nr}}\ B$;
- $n \ \underline{\mathsf{nr}}\ (A \to B)$ if for all m, s.t. $m \ \underline{\mathsf{nr}}\ A$, partial recursive function $\{n\}$ is defined at m and $\{n\}m \ \underline{\mathsf{nr}}\ B$;
- $n \ \underline{\mathsf{nr}}\ \forall x.A(x)$ if for all $m \in \mathbf{N}$, partial recursive function $\{n\}$ is defined at m and $\{n\}m \ \underline{\mathsf{nr}}\ A(\bar{m})$;
- $n \ \underline{\mathsf{nr}}\ \exists x.A(x)$ if $\mathsf{p}_2 n \ \underline{\mathsf{nr}}\ A(\overline{\mathsf{p}_1 n})$.

Here $\{-\}-$ denotes Kleene application, p_1 and p_2 denote primitive recursive projection functions , and $\bar{m}$ is the representation of number m inside $\mathbf{HA}$. We say that a sentence φ is realized, if there is a number n such that $n \ \underline{\mathsf{nr}}\ \varphi$.

The number realizability enjoys a number of useful properties.

Theorem 4.1 (Soundness)**.** *If* $\mathbf{HA} \vdash \varphi$*, then there is a number* n *such that* $n \ \underline{\mathsf{nr}}\ \varphi$*.*

Proof. Induction on derivation of φ. □

As an immediate consequence,

Example 4.2 (Consistency of **HA**)**.** If $\mathbf{HA} \vdash \bot$, then $\bot$ is realized, which is clearly impossible.

Example 4.3 (Weak disjunction property)**.** If $\mathbf{HA} \vdash \varphi \vee \psi$, then there is a number n, such that either $n \ \underline{\mathsf{nr}}\ \varphi$ or $n \ \underline{\mathsf{nr}}\ \psi$.

The number realizability provides an interesting interpretation of Heyting arithmetic, as it validates a number of constructive non-classical principles; we refer the reader to e.g. [38, 41] for details.

In this chapter, we present the effective topos [22] – a topos in which arithmetic coincides with the realizability interpretation, and which logic can be seen as a generalization of realizability to higher-order topos logic.

4.2 Realizability logic

Before we embark on the description of the effective topos, we would like to fix what we call "realizability notation". For each first-order formula φ involving variables and constants ranging over $\mathcal{P}(\omega)$ we assign an element of $\mathcal{P}(\omega)$.

- If P is a subset of natural numbers, then the interpretation $[P]$ of P is P itself. We also put $[\top] = \mathbb{N}$, $[\bot] = \emptyset$.
- If φ and ψ are formulas, then $[\varphi \wedge \psi] = [\varphi] \wedge [\psi] = \{\langle n, m\rangle \mid n \in [\varphi], m \in [\psi]\}$.
- If φ and ψ are formulas, then $[\varphi \vee \psi] = [\varphi] \vee [\psi] = \{\langle 0, n\rangle \ \mid n \in [\varphi]\} \cup \{\langle 1, m\rangle \mid m \in [\psi]\}$
- If φ and ψ are formulas, then $[\varphi \to \psi] = [\varphi] \to [\psi] = \{e \in \omega \mid \forall x \in [\varphi], e \cdot x \in [\psi]\}$.
- If φ is a formula with a variable x, then $[\exists x : X.\varphi(x)] = \bigcup_{x \in X}[\varphi(x)]$.

- If φ is a formula with a variable x, then $[\forall x : X.\varphi(x)] = \{e \in \omega \mid \forall n \in \omega \forall x \in X(e \cdot n\downarrow \wedge e \cdot n \in [\varphi(x)])\}$.

In particular, the quantifiers are interpreted in a *uniform* way, which corresponds to a sort of polymorphism in programming languages. This can be contrasted with a Curry-Howard style correspondence, in which quantifiers are interpreted in a non-uniform way: $\forall x : X.\varphi(x)$ would be interpreted as a type $\Pi_{x:X}\varphi(x)$; when constructing a term of such type, one could appeal to the "evidence" that x is of type X. We will see later how to non-uniform quantification in the effective topos arises as standard interpretation of logic in topoi.

If $n \in [\varphi]$, we say that n *realizes* φ (later we will see that this "confusion" of terminology is warranted).

Example 4.4. The set $[\mathbb{N} \to \mathbb{N}]$ contains exactly indices of total recursive functions $\mathbb{N} \to \mathbb{N}$. In general, if $D \subseteq \mathbb{N}$, then $[D \to \mathbb{N}]$ contains indices of partial recursive functions which are defined on D.

Example 4.5. If φ and ψ are sentences of Heyting arithmetic, and R_φ and R_ψ are sets of realizers for φ and ψ respectively, then $[R_\varphi \wedge R_\psi]$ is a set of realizers for $\varphi \wedge \psi$.

Example 4.6. If $E : \mathbb{N} \to \mathcal{P}(\omega)$ is a predicate on natural numbers such that $E(n) = \{1\}$ if n is even and $E(n) = \{0\}$ if n is odd, then $[\forall x : \mathbb{N}(\{n\} \to E(n))]$ is a set of indices of recursive functions ϕ, such that ϕ decides the set $\{0, 2, 4, \dots\}$ of even numbers. For if $e \in [\forall n : \mathbb{N}(\{n\} \to E(n))]$ is an index of a recursive function $\{e\}(-) = \phi$, then ϕ must be *uniform*, i.e. for every $n \in \mathbb{N}$, ϕ must be defined on $\{n\}$, and $\phi(n)$ have to be in $E(n)$. To say, $\phi(n)$ must take values in $\{0, 1\}$ and it computes to 1 iff n is even.

For $A, B \in \mathcal{P}(\omega)$, we write $A \vdash B$ (or $A \leq B$) if there is a recursive function φ, such that $\varphi \cdot e \in B$ if $e \in A$. One can see that $(\mathcal{P}(\omega), \vdash)$ forms a Heyting prealgebra w.r.t operations defined above.

4.3 Basic definitions and examples

In this section we describe the category $\mathcal{E}ff$ – the effective topos. The objects of $\mathcal{E}ff$ are pairs $(X, \sim)$, where $\sim$ is a function $X \times X \to \mathcal{P}(\omega)$, usually written as $x, y \mapsto [x \sim y]$, such that there are recursive functions s and tr satisfying

1. If $a \in [x \sim y]$, then $\mathsf{s}(a) \in [y \sim x]$;
2. If $a \in [x \sim y]$ and $b \in [y \sim z]$, then $\mathsf{tr}(a, b) \in [x \sim z]$.

Equivalently, using the notation from the previous section we may write

- $\mathsf{s} \in [\forall x : X, y : X([x \sim y] \to [y \sim x])]$
- $\mathsf{tr} \in [\forall xyz : X([x \sim y] \wedge [y \sim z] \to [x \sim z])]$

We think of $[x \sim x]$ as a set of "realizers" for the "existence" of $x \in X$, and we sometimes denote it by $E_X(x)$.

Given two objects $(X, \sim_X)$ and $(Y, \sim_Y)$ of the effective topos, a map between them as a relation $F : X \times Y \to \mathcal{P}(\omega)$, satisfying the following conditions:

(REL) Given $n \in [x \sim_X x']$, $m \in [y \sim_Y y']$ and $p \in F(x, y)$, one can recursively and uniformly find an element $\mathsf{rel}_F(n, m, p) \in F(x', y')$

(ST) Given $n \in F(x, y)$ one can recursively find $\mathsf{st}_X(n) \in [x \sim_X x]$ and $\mathsf{st}_Y(n) \in [y \sim_Y y]$

(SV) Given $n \in F(x, y)$ and $m \in F(x, y')$, one can recursively find $\mathsf{sv}_F(n, m) \in [y \sim_Y y']$

(TOT) Given $n \in [x \sim_X x]$, one can recursively find $\mathsf{tot}_F(n) \in \bigcup_{y \in Y} F(x, y)$

We say that relation F is *relational* (or, respects $\sim_X$ and $\sim_Y$), *strict*, *single-valued*, and *total*.

Two maps $F, G : (X, \sim_X) \to (Y, \sim_Y)$ are equal iff there is a computable function $\psi \in [\forall xy.F(x, y) \to G(x, y)]$ (written as $F \leq G$) and, symmetrically, $\phi \in [\forall xy.G(x, y) \to F(x, y)]$ (written as $G \leq F$).

The identity map on $(X, \sim)$ is represented by the relation $\sim$ itself. The composition of two maps $F : (X, \sim) \to (Y, \approx)$ and $G : (Y, \approx) \to (Z, \smile)$ is a relation

$$(G \circ F)(x, z) = [\exists y : Y.F(x, y) \wedge G(y, z)]$$

We leave it to the reader to check that this in fact determines a morphism in $\mathcal{E}ff$, and that the composition operation satisfies the usual categorical laws.

Remark 4.7 (When two morphisms are equal?)**.** Surprisingly, to show that $F = G$ it suffices to check that $F \leq G$! For the other direction, suppose $G(x, y)$; then $[x \sim_X x]$. By the totality of F, we have $F(x, y_0)$ for some y_0. But because $F \leq G$, it is the case that $G(x, y_0)$. Due to single-valuedness of G we have $[y \sim_Y y_0]$. Finally, by relationality of F, $F(x, y)$.

Remark 4.8 (When a map is an isomorphism?)**.** Suppose we are given a morphism $G : (X, \sim_X) \to (Y, \sim_Y)$ as a functional relation. We can always consider an opposite relation G^{-1}. If G^{-1} itself a morphism, then the composite $G \circ G^{-1}$ is the identity on $(Y, \sim_Y)$:

$$(G \circ G^{-1})(y, y') = \bigcup_{x \in X} G^{-1}(y, x) \wedge G(x, y') = \bigcup_{x \in X} G(x, y) \wedge G(x, y')$$

which implies, by single-valuedness, that $y \sim_Y y'$. On the other hand, if $n \in [y \sim_Y y']$, then there always exists, by totality, an $x \in X$, such that $\phi(n) \in G^{-1}(y, x) = G(x, y)$. Then, by relational property, $\psi(n, \phi(n)) \in G(x, y')$. Thus $(G \circ G^{-1}) =\sim_Y$ and similarly $(G^{-1} \circ G) =\sim_X$, if only G^{-1} is a morphism as well. But when is it a morphism? The relational property and strictness of G^{-1} follows directly from the corresponding properties of G. It is thus sufficient to check that G^{-1} is total and single-valued in order to establish that G is an isomorphism.

Set-level induced maps. Given two objects $(X, \sim)$ and $(Y, \approx)$ of $\mathcal{E}ff$, a set-level function $f : X \to Y$ can *induce* a morphism in $\mathcal{E}ff$. If there is a recursive ψ, s.t. $a \in [x \sim y] \implies \psi(a) \in [f(x) \approx f(y)]$, then the induced a morphism $F : (X, \sim) \to (Y, \approx)$ given by

$$F(x, y) = \bigcup_{x' \in X} \{\langle a, b \rangle \mid a \in [x \sim x'], b \in [f(x') \approx y]\}$$

Effective topos and realizability. The connection between Kleene's realizability and the effective topos might not be evident at this point. One way to see this would be to examine the *tripos* construction of the effective topos (see [32, 23] and [29, Chapter 2] for details). The effective topos then arises from a realizability tripos for Kleene's first algebra. The internal logic of that tripos, as a hyperdoctrine, is the realizability logic of section 4.2.

Effective topos, as any elementary topos, possesses an interpretation of higher-order logic. As it turns out, this interpretation can be reduced to realizability logic in a way similar to reduction of uniform quantifiers to non-uniform ones. But we shall delay this reduction until we describe subobjects in $\mathcal{E}ff$.

4.4 Some constructions in the effective topos

Terminal object. The terminal object $(1, =_1)$ is given by a one-element set, with the realizability relation $[* =_1 *] = \omega$. Suppose that $(X, \sim)$ is an object of the effective topos. Then there is an evident morphism $!_X : (X, \sim) \to (1, =_1)$ (written as ! whenever unambiguous), as defined by

$$!(x, *) = [x \sim x]$$

It is easy to verify that ! satisfies the required properties to be a morphism. Furthermore, suppose that $F : (X, \sim) \to (1, =_1)$ is another morphism. We can prove that $F =!$ as follows

($F \leq !$) Given $n \in F(x, *)$ we can, by strictness, obtain $\psi(n) \in [x \sim x] =!(x, *)$.

($! \leq F$) Given $m \in !(x, *) = [x \sim x]$, we can obtain, by totality, $\rho(m) \in \bigcup_{y \in 1} F(x, y) = F(x, *)$.

Products. Given $(X, \sim_X)$ and $(Y, \sim_Y)$ the product $(X, \sim_X) \times (Y, \sim_Y)$ is a object $(X \times Y, \sim)$, where the realizability equality is defined as

$$(x, y) \sim (x', y') = (x \sim_X x') \wedge (y \sim_Y y')$$

The first projection F_{π_1} is defined as $F_{\pi_1}((x, y), x') = [x \sim_X x'] \wedge [y \sim_Y y]$. Similarly, the second project is given by $F_{\pi_2}((x, y), y') = [x \sim_X x] \wedge [y \sim_Y y']$. It is straightforward to verify that those relations satisfy the required properties to be morphisms.

Given an object $(A, \approx)$ and maps $F : (A, \approx) \to (X, \sim_X)$, $G : (A, \approx) \to (Y, \sim_Y)$, there is a unique map $\langle F, G \rangle : (A, \approx) \to (X \times Y, \sim)$, given by

$$\langle F, G \rangle (a, (x, y)) = F(a, x) \wedge G(a, y)$$

Then

$$(F_{\pi_1} \circ \langle F, G \rangle)(a, x) = \bigcup_{(x', y) : X \times Y} F(a, x') \wedge G(a, y) \wedge [x \sim_X x'] \wedge [y \sim_Y y]$$

Hence from $(F_{\pi_1} \circ \langle F, G \rangle)(a, x)$ you can extract $F(a, x)$ (via $F(a, x')$ and $x \sim_X x'$). Conversely, given $F(a, x)$ you can find an element of $[a \approx a] \wedge [x \sim_X x]$, and, hence $G(a, y)$ for some y. From this information you can obtain an element of $(F_{\pi_1} \circ \langle F, G \rangle)(a, x)$. Similarly for F_{π_2} and G.

Equalizers. Given a parallel pair of maps $F, G : (X, \sim_X) \to (Y, \sim_Y)$, an equalizer for F and G is a subobject E of $(X, \sim_X)$ defined to be $(X, \approx)$ where

$$x \approx x' = x \sim_X x' \wedge \exists y : Y(F(x, y) \wedge G(x, y))$$

(in the terminology of next section, E is a subobject of $(X, \sim_X)$ determined by the predicate $P(x) = \exists y : Y(F(x, y) \wedge G(x, y))$). The inclusion $(X, \approx) \rightarrowtail (X, \sim_X)$ is represented by $\approx$ itself. We leave it to the reader to check that $\approx$ satisfies the requirements for a morphism in $\mathcal{E}ff$.

Given an element of

$$(F \circ \approx)(x, y) = \exists x' : X.x \approx x' \wedge F(x', y) = \exists x'.x \sim_X x' \wedge \exists y' : Y(F(x, y') \wedge G(x, y')) \wedge F(x'y)$$

we can obtain an element of

$$(G \circ \approx)(x, y) = \exists x'.x \sim_X x' \wedge \exists y' : Y(F(x, y') \wedge G(x, y')) \wedge G(x'y)$$

by using the same x' as a witness, and employing relationality and single-valuedness of G. Thus $F \circ \approx = G \circ \approx$.

Suppose $M : (C, -) \to (X, \sim_X)$ is another arrow equalizing F and G. We prove that M restricts to $(X, \approx)$.

(REL) If $c - c' \wedge M(c, x) \wedge x \approx x'$, then $M(c, x')$ because M is relational w.r.t. $\sim_X$ and $x \approx x'$ implies $x \sim_X x'$.

(TOT) The same as totality w.r.t. $(X, \sim_X)$.

(ST) If $M(c, x)$, then $c - c$ and $x \sim_X x$ by strictness w.r.t. $(X, \sim_X)$. By totality of F we have $F(x, y)$ for some y. Since $(F \circ M)(c, y) \simeq (G \circ M)(c, y)$ and we have $M(c, x) \wedge F(x, y)$, we obtain $M(c, x') \wedge G(x', y)$. By single-valuedness of M w.r.t. $\sim_X$, we get $x \sim_X x'$, and, hence $G(x, y)$. Thus $F(x, y) \wedge G(x, y)$. So $x \approx x$.

(SV) If $M(c, x) \wedge M(c, x')$, then $x \sim_X x'$ by single-valuedness w.r.t. $\sim_X$; by strictness, $x \approx x$, and hence $x \approx x'$.

Pullbacks. Given internal descriptions of products and equalizers, we can construct a pullback of $G : (Y, \sim_Y) \to (Z, \sim_Z)$ and $F : (X, \sim_X) \to (Z, \sim_Z)$ as $(X \times_Z Y, \sim)$ where

$$(x, y) \sim (x', y') = x \sim_X x' \wedge y \sim_Y y' \wedge \exists z : Z(F(x, z) \wedge G(y, z))$$

An intensional logical definition of a pullback is also possible.

Proposition 4.9. *A commuting diagram*

$$\begin{array}{ccc} (P, \sim) & \xrightarrow{H} & (Y, \sim_Y) \\ \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle G} \\ (X, \sim_X) & \xrightarrow{F} & (Z, \sim_Z) \end{array}$$

is a pullback if and only if

$$\begin{cases} \forall xypp'.H(p, y) \wedge K(p, x) \wedge H(p', y) \wedge K(p', x) \to p \sim p' \\ \forall xyz.(F(x, z) \wedge G(y, z) \to \exists p(H(p, y) \wedge K(p, x))) \end{cases}$$

Proof. ($\Rightarrow$) If $(P, \sim)$ is a pullback, it is isomorphic to $(X \times_Z Y, \sim)$, the isomorphism witnessed by Φ. We want to verify two conditions above.

Suppose that $H(p, y) \wedge K(p, x) \wedge H(p', y) \wedge K(p', x)$. Then, since the square commutes, we obtain $G(y, c) \wedge F(x, c)$. Hence we have $[(x, y) \sim (x, y)]$ in $(X \times_Z Y, \sim)$. Thus, by totality of Φ^{-1} we have $\Phi(p_0, x, y)$ for some $p_0 \in P$. On the other hand, we have $\Phi(p, x_0, y_0)$ by the totality of Φ. Since $P_1 \circ \Phi \simeq K$ and $P_2 \circ \Phi \simeq H$, we have $(P_1 \circ \Phi)(p, x_0) \simeq K(p, x_0)$ and by single-valuedness of K we obtain $x \sim_X x_0$. Similarly, $y \sim_Y y_0$. By the single-valuedness of Φ^{-1} we have $p_0 \sim p$; but we can repeat the same argument with p' instead of p, it follows that $p \sim p'$.

For the second condition suppose that $F(x, z) \wedge G(y, z)$. Then $[(x, y) \sim (x, y)]$ in $(X \times_Z Y, \sim)$; we obtain the desired p from the totality of Φ^{-1}.

($\Leftarrow$) Suppose on the other hand that $(P, \sim)$ satisfies the properties above. Consider a map $M : (P, \sim) \to (X \times_Z Y, \sim)$ defined as

$$M(p, x, y) = H(p, y) \wedge K(P, x)$$

One can check that this is well-defined map. By the discussion in remark 4.8, to check that M is an isomorphism is to check that M^{-1} is single-valued and total – but those are exactly the two conditions we put on $(P, \sim)$.

□

Natural numbers object. The natural numbers object $\mathbf{N}$ is given by $(\mathbb{N}, \sim)$, where $[n \sim m] = \{n\}$ if $n = m$ and $[n \sim m] = \emptyset$ otherwise. See [22, Proposition 3.2] for the proof that $\mathbf{N}$ is indeed a natural numbers object.

Constant objects functor. Of importance is the "constant objects" functor ∇ : **Set** $\to \mathcal{E}ff$, which is the right adjoint to the global sections functor $\Gamma : \mathcal{E}ff \to$ **Set**, $\Gamma(X) = Hom_{\mathcal{E}ff}(1, X)$. Explicitly, $\nabla(A)$ is an object $(A, \sim)$ with $[a \sim a] = \omega$ and $[a \sim b] = \emptyset$ if $a \neq b$. For a map $f : A \to B$ the map $\nabla(f)$ is induced by f on the level of sets, and is tracked by $\lambda x.x$.

Remark 4.10. Using the definition of ∇ above, we can say that the terminal object $(1, =_1)$ is actually $\nabla 1$.

Coequalizers of equivalence relations. Since $\mathcal{E}ff$ is a topos, all coequalizers exist in $\mathcal{E}ff$. However, it is not easy to calculate arbitrary coequalizers in such a canonical way.

First of all, if $(X \times X, \sim_P) \rightarrowtail (X \times X, \sim)$ is an equivalence relation on X, given by a strict predicate P as a subobject of $X \times X$, we can compute the coequalizer of

$$(X \times X, \sim_P) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} (X, \sim)$$

as an object $(X, \approx)$ as $[x \approx y] = P(x, y)$. Symmetry and transitivity is obtained from symmetry and transitivity of P. The arrow $(X, \sim) \to (X, \approx)$ is represented by P itself.

Given any other object $(Y, \sim_Y)$ and an arrow $F : (X, \sim) \to (Y, \sim_Y)$, equalizing π_1 and π_2, we have a unique $H : (X, \approx) \to (Y, \sim_Y)$

$$\begin{array}{ccccc} (X \times X, \sim_P) & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & (X, \sim) & \xrightarrow{P} & (X, \approx) \\ & & & \searrow^{F} & \downarrow{\scriptstyle H} \\ & & & & (Y, \sim_Y) \end{array}$$

determined by $H(x, y) = F(x, y)$. It is a well-defined morphism:

(REL) $[x_1 \approx x_2] \wedge [y_1 \sim_Y y_2] \wedge F(x_1, y_1)$ implies $F(x_1, y_2)$. Furthermore, because F equalizes π_1 and π_2, we have a function sending

$$(F \circ \pi_1)(x_1, x_2, y) = \exists x.P(x_1, x_2) \wedge [x_1 \sim x] \wedge F(x, y)$$

to

$$(F \circ \pi_2)(x_1, x_2, y) = \exists x.P(x_1, x_2) \wedge [x_2 \sim x] \wedge F(x, y)$$

Then take $(F \circ \pi_1)(x_1, x_2, y) \simeq P(x_1, x_2) \wedge F(x_1, y_2)$ and apply the function to obtain $F(x_2, y_2)$.

(ST) Given $F(x, y)$ we get $[x \sim x]$ and $[y \sim_Y y]$ we have $P(x, x) = [x \approx x]$, since P is reflexive.

(SV) Follows from the single-valuedness of F

(TOT) Given $[x \approx x]$ one has $[x \sim x]$, by the stability of P; the rest follows by the totality of F.

Arbitrary coequalizers. To compute an arbitrary coequalizer

$$(A, \sim_A) \overset{F}{\underset{G}{\rightrightarrows}} (B, \sim_B) \overset{q}{\twoheadrightarrow} (B, \approx)$$

we first obtain the smallest equivalence relation through which $\langle f, g \rangle$ factors, after which we apply the method described in a previous paragraph. Such an equivalence relation is a subobject $(B \times B, \sim) \hookrightarrow (B \times B, \sim_{B \times B})$ which is determined by

$$P(b_0, b_1) \iff [b_0 \sim b_1]$$
$$\wedge \forall A : \mathcal{P}(\omega)^{B\times B}.[Equiv(A) \wedge (\forall b, b' : B, a : A.F(a, b) \wedge G(a, b') \to R(b, b'))] \to R(b_0, b_1)$$

where $Equiv(A) = Rel(A) \wedge Refl(A) \wedge Sym(A) \wedge Tr(A)$ states that A is an equivalence relation on B:

1. $Rel(A) := \forall x, y : B.(R(x, y) \to [x \sim_B x] \wedge [y \sim_B y]) \wedge \forall x', y' : B.([x \sim_B x'] \wedge [y \sim_B y'] \wedge R(x, y) \to R(x', y'))$
2. $Refl(A) := \forall x x' : B.[x \sim_B x'] \to R(x, x')$
3. $Sym(A) := \forall x, y : B.R(x, y) \to R(y, x)$
4. $Tr(A) := \forall x, y, z : B.R(x, y) \wedge R(y, z) \to R(x, z)$

We can then verify that $(B, \approx)$ is the coequalizer. First of all, to show that $(\approx \circ F) \simeq (\approx \circ G)$, suppose that $(\approx \circ F)(a, b) = \exists b_0 : B.F(a, b_0) \wedge [b_0 \approx b]$. Then, by totality of G, we have $G(a, b_1)$ for some b_1. Thus, any R through which $\langle F, G\rangle$ factors contains $R(b_0, b_1)$. We have $R(b_0, b)$ by assumption; by symmetry and transitivity of R, we have $R(b_1, b)$.

Secondly, if the map $K : (B, \sim_B) \to (Y, \sim_Y)$ coequalizes F, G, then we can define a map $H : (B, \approx) \to (Y, \sim_Y)$ as $H(b, y) = K(b, y)$. It is a well-defined morphism:

(REL) Given $[b \approx b'] \wedge H(b, y)$, we want to obtain $H(b', y)$. Take an equivalence relation $R(b, b') = [b \sim_B b] \wedge [b' \sim_B b'] \wedge (K(b, y) \leftrightarrow K(b', y))$. Clearly, R is a reflexive, symmetric and transitive relation. We want to show that $\langle F, G\rangle$ factors through R; for that suppose that $F(a, b) \wedge G(a, b')$ for some a, b, b'. Furthermore, suppose w.l.o.g. $K(b, y)$. Then we have $(K \circ F)(a, y)$; because $K \circ F \simeq K \circ G$ we can obtain $(K \circ G)(a, y)$, i.e. $G(a, b_1) \wedge K(b_1, y)$ for some b_1; by single-valuedness of G, we have $[b' \sim_B b_1]$ and hence $K(b', y)$.

(ST) Given $K(b, y)$ we can get $[b \approx b]$ by reflexivity.

(SV) By single-valuedness of K.

(TOT) By the totality of K.

4.5 Subobjects and the subobject classifier

Subobjects in $\mathcal{E}ff$. We start by describing the canonical presentation of subobjects in $\mathcal{E}ff$. We say that a predicate G on Y (that is, $G : Y \to \mathcal{P}(\omega)$) is *strict* (with regard to a realizability relation $\sim_Y$), if there are realizers for the following statements

1. $\forall y : Y.G(y) \to [y \sim_Y y]$
2. $\forall y, y' : Y.G(y) \wedge [y \sim_Y y'] \to G(y')$

A subobject $(Y', \approx_G)$ of $(Y, \sim_Y)$ is in canonical form iff

- $Y' = Y$
- $x \approx_G x' = [x \sim_Y x'] \wedge G(x)$ for a strict predicate G

One can easily check that for a strict G, the relation $\approx_G$ is recursively symmetric and transitive.

Proposition 4.11. *Every subobject is isomorphic to a subobject in canonical form.*

Proof. Given $(X, \sim_X) \rightarrowtail (Y, \sim_Y)$ represented by $F : X \times Y \to \mathcal{P}(\omega)$, define $G(y) = \exists x : X.F(x, y)$. Then G is a strict predicate due to strictness and relationality of F.

We define a morphism $H : (X, \sim_X) \to (Y, \approx_G)$ by

$$H(x, y) = F(x, y)$$

To show that H is an isomorphism, we need to show that $H^{-1} = F^{-1}$ is single-valued and total (w.r.t. $\approx_G$). The single-valuedness of F^{-1} follows from the fact that it is a mono. To see that it is total, suppose that $[y \approx_G y] = [y \sim_Y y] \wedge G(y) = [y \sim_Y y] \wedge \exists x : X.F(x, y)$. From that we can clearly extract $\exists x : X.F^{-1}(y, x)$. $\square$

As we have already seen, there is a recursion-theoretic presentation of general pullbacks in $\mathcal{E}ff$. However, in case of subobjects it can be more convenient to use the characterization below.

Proposition 4.12. *Suppose that $(X, \approx_G) \rightarrowtail (X, \sim_X)$ is a subobject of $(X, \sim_X)$ determined by a predicate G, and $F : (Y, \sim_Y) \to (X, \sim_X)$ is an arrow. Then the pullback of $(X, \approx_G)$ along F can be described as a subobject of $(Y, \sim_Y)$ determined by the predicate G'*

$$G'(y) \equiv \exists x.F(y, x) \wedge G(x)$$

Proof. The map $F' : (Y, \approx_{G'}) \to (X, \approx_G)$ is defined as $F'(y, x) = F(y, x) \wedge G'(y) \wedge G(x)$. It is straightforward to verify that the resulting square commutes, hence we can apply the formula from proposition 4.9.

1. $F'(y, x) \wedge [y \approx_{G'} y_0] \wedge F'(y', x) \wedge [y' \approx_{G'} y_0] \to [y \approx_{G'} y]$; follows from the transitivity of $\approx_{G'}$.

2. $F(y, x) \wedge [x_0 \approx_G x] \to \exists y'.F'(y', x) \wedge [y' \approx_{G'} y]$; take $y' = y$, the rest follows by the definition of G'.

$\square$

Subobject classifier. Recall that for $A, B \in \mathcal{P}(\omega)$, we denote by $[A \to B]$ a set of codes of recursive functions ϕ, such that whenever $a \in A$, $\phi(a) \in B$. Denote by $[A \leftrightarrow B]$ the set $[A \to B] \wedge [B \to A]$.

Proposition 4.13. *The subobject classifier Ω in the effective topos is given by $(\mathcal{P}(\omega), \leftrightarrow)$ with the truth arrow $t : (1, =_1) \to (\mathcal{P}(\omega), \leftrightarrow)$ given by $t(*, A) := [A \leftrightarrow \omega]$.*

Proof. We can easily check that t, defined as above, is a morphism in $\mathcal{E}ff$. It remains to show that Ω does really classify subobjects. Let $(X, \approx_G) \rightarrowtail (X, \sim)$ be a subobject given by a strict predicate G, we define $\chi_G : (X, \sim) \to \Omega$ as

$$\chi_G(x, A) = [x \sim x] \wedge [G(x) \leftrightarrow A]$$

We must verify that

$$\begin{array}{ccc} (X, \approx_G) & \longrightarrow & (1, =_1) \\ \downarrow & & \downarrow t \\ (X, \sim) & \xrightarrow{\chi_G} & (\mathcal{P}(\omega), \leftrightarrow) \end{array} \tag{4.1}$$

is a pullback. For that we utilize the formula from proposition 4.12 for pullbacks of subobjects in canonical form. We can view $(1, =) \rightarrowtail \Omega$ as a subobject given by the relation $P(A) = [A \leftrightarrow \omega]$. Then the pullback of $(1, =_1) \simeq (\mathcal{P}(\omega), \approx_P) \rightarrowtail (\mathcal{P}(\omega), \leftrightarrow)$ is given by the strict predicate

$$K(x) = \exists A : \mathcal{P}(\omega)(P(A) \wedge \chi_G(x, A)) = \exists A : \mathcal{P}(\omega)([A \leftrightarrow \omega] \wedge [x \sim x] \wedge [G(x) \leftrightarrow A])$$

We clearly have $K(x) \leftrightarrow G(x)$, so $(X, \approx_G) \simeq (X, \approx_K)$, hence diagram (4.1) is a pullback. It remains to show that χ_G is unique. Suppose that $F : (X, \sim) \to (\mathcal{P}(\omega), \leftrightarrow)$ is another arrow making (4.1) a pullback. Then, according to the logical description of a pullback in $\mathcal{E}ff$

$$F(x, A) \wedge t(*, A) \vdash \exists x' : X(x' \approx_G x) \wedge !(x', *)$$

Thus, given $n \in F(x, A)$ and $a \in A$, we can recursively obtain $\psi(n, a) \in [x' \sim x] \wedge G(x')$ for some x'. Clearly, from that we can get $G(x)$. Hence $A \to G(x)$, given some $n \in F(x, A)$.

Additionally, we have $F \circ \approx_G = t \circ !$; that is, given

$$(F \circ \approx_G)(x, A) = \exists x' : X.x \approx_G x' \wedge F(x', A)$$

we can obtain

$$(t \circ !)(x, A) = !(x, *) \wedge [A \leftrightarrow \omega]$$

That is, given $n \in F(x, A)$ and $k \in G(x)$, we can obtain $[A \leftrightarrow \omega]$, and, hence some $a \in A$. Thus $G(x) \to A$, given some $n \in F(x, A)$. Hence, we can conclude that $F(x, A) \to [G(x) \leftrightarrow A]$, so $F \leq \chi_G$. □

4.6 Logic in $\mathcal{E}ff$

In this section we describe the internal logic in $\mathcal{E}ff$ in terms of realizability notation of section 4.2. This will allow us to check the formulas of topos logic for validity in $\mathcal{E}ff$ by translating them to realizability logic. For a more general overview overview, a reader is directed to [29, Section 2.3].

Heyting algebra of subobjects. As in every topos, the collection $Sub(X, \sim)$ of subobjects of $(X, \sim)$ forms a Heyting algebra. Below we give explicit definitions of each operation.

Suppose that $(X, \sim_P)$ and $(X, \sim_Q)$ are subobjects of $(X, \sim)$ in canonical form, induced by strict predicates P and Q resp. Then,

- The subobject $(X, \sim_P) \wedge (X, \sim_Q) = (X, \sim_{P \wedge Q})$ is induced by a strict predicate $(P \wedge Q)(x) = [P(x) \wedge Q(x)]$.
- The subobject $(X, \sim_P) \vee (X, \sim_Q) = (X, \sim_{P \vee Q})$ is induced by a strict predicate $(P \vee Q)(x) = [P(x) \vee Q(x)]$.
- As usual, the subobject $\top$ is $(X, \sim)$ itself, and $\bot$ is the initial object $\emptyset$.
- The subobject $(X, \sim_P) \to (X, \sim_Q) = (X, \sim_{P \to Q})$ is induced by a strict predicate $(P \to Q)(x) = [[x \sim x] \wedge (P(x) \to Q(x))]$ (the predicate $(P(x) \to Q(x))$ is not strict by itself, so we need $x \sim x$ as an additional piece of information).

In particular, the negation $\neg P = P \to \bot$ of a strict predicate P determines a subobject of $(X, \sim)$ that only contains elements of X for which $P(x)$ is empty, with no additional information. Similarly, doubly-negating a subobject $(X, \sim_P)$ determines a subobject of X containing only elements for which $P(x)$ is non-empty, but all the additional information provided by P is erased.

Quantifiers. The quantification is interpreted as follows: if $\varphi = (X \times Y, \sim_P)$ is a subobject of $(X, \sim) \times (Y, \approx)$ determined by a strict predicate P, then

- $\exists x : X\varphi(x, y)$ is interpreted as a subobject of $(Y, \approx)$ determined by a strict predicate $(\exists x : X.P)(y) = [\exists x : X.[x \sim x] \wedge P(x, y)]$.
- $\forall x : X\varphi(x, y)$ is interpreted as a subobject of $(Y, \approx)$ determined by a strict predicate $(\forall x : X.P)(y) = [[y \approx y] \wedge \forall x : X.[x \sim x] \to P(x, y)]$.

This reduction of quantification in $\mathcal{E}ff$ to quantification in realizability logic is reminiscent of interpretation of non-uniform quantifiers in terms of uniform quantifiers. Note that in the $\exists$ case, one can actually obtain $[x \sim x]$ from $P(x, y)$, so we could have put $(\exists x : X.P)(y) = [\exists x : X.P(x, y)]$, which is equivalent.

Equality. In a topos, the equality predicate on an object X is defined to be the characteristic map of the diagonal inclusion $\delta_X : X \to X \times X$. In the effective topos, the diagonal subobject of $(X, \sim) \times (X, \sim)$ is represented by $Eq_X(x, y) = [x \sim y]$. Thus, in formulas of topos logic ...$x = y$... can be rewritten to ...$[x \sim y]$..., where x and y range over an object $(X, \sim)$.

Powersets. Given an object $(X, \sim)$ of $\mathcal{E}ff$, the powerset $\mathcal{P}(X, \sim)$ (or $\mathcal{P}(X)$ for short) is defined as the exponent $(\Omega^X, \Leftrightarrow)$. We can actually give a simplified explicit description of that object as follows.

The underlying set of $\mathcal{P}(X)$ is a set of strict relations on X, i.e. maps $X \to \mathcal{P}(\omega)$. The realizability predicate is defined as

$$P \Leftrightarrow Q := [st(P) \wedge \forall x : X.[P(x) \leftrightarrow Q(x)]]$$

where $st(P)$ says that P is a strict predicate:

$$st(P) := [\forall x : X.(P(x) \to [x \sim x]) \wedge \forall x, y : X.(P(x) \wedge [x \sim y] \to P(y))]$$

Then, the membership predicate $\in_X \hookrightarrow (X, \sim) \times \mathcal{P}(X, \sim)$ is defined by a strict predicate $(x, P) \mapsto st(P) \wedge P(x)$. Thus, for instance, a formula $\forall x : X \exists P : \mathcal{P}(X).x \in P$ is interpreted as $[\forall x : X.[x \sim x] \to \exists P : \mathcal{P}(X).(st(P) \wedge P(x))]$ and is realized by, e.g. $\lambda n.\langle m, n\rangle$ where m realizes the fact that $y \mapsto [x \sim y]$ is a stable predicate.

First-order arithmetic in $\mathcal{E}ff$. Consider a formula $\varphi(x_1, \ldots, x_n)$ of first-order arithmetic, with $x_1, \ldots, x_n$ ranging over natural numbers. It is interpret in $\mathcal{E}ff$ as a map $\mathbf{N} \times \mathbf{N} \times \cdots \times \mathbf{N} \to \Omega$, or, equivalently, as a subobject of $\mathbf{N}^k$. Thus, it corresponds to some strict predicate P on $\mathbf{N}^k$. One can then prove, by induction on φ, the following correspondence/completeness theorem:

Theorem 4.14. *[29, Proposition 3.1.3] There are primitive recursive functions α_φ and β_φ, such that*

- *If $a \in P(m_1, \ldots, m_n)$, then $\alpha_\varphi(a)$ $\underline{\mathsf{nr}}$ $\varphi(m_1, \ldots, m_n)$;*
- *If b $\underline{\mathsf{nr}}$ $\varphi(m_1, \ldots, m_n)$, then $\beta_\varphi(b) \in P(m_1, \ldots, m_n)$.*

Thus, arithmetic in $\mathcal{E}ff$ is precisely captured by the Kleene realizability semantics of Heyting arithmetic. This reasoning can be generalized to second-order arithmetic and Troelstra realizability, see [29, Section 3.1.1] for details.

It follows, by the definition of $\mathbf{N}$, that the morphisms $\mathbf{N} \to \mathbf{N}$ in the effective topos are precisely the computable functions $\mathbb{N} \to \mathbb{N}$. In fact, morphisms between all objects built out of $\mathbf{N}$ are computable (in a higher sense); see [22, Section 11] for a formal statement.

4.7 Some subcategories and classes of objects in the effective topos

Assemblies. Effective topos can be seen as a "universe" for constructive mathematics in a sense that $\mathcal{E}ff$ validates (as a topos) exactly those formulas of higher-order logic that are realizable (for a suitable higher-order generalization of realizability). However, if one does not need the whole power of higher-order logic, one can restrict attention to a subcategory of *assemblies*, which admits a somewhat easier definition.

Categorically, assemblies are $\neg\neg$-separated objects in the following sense:

Definition 4.15. An object $(X, \sim)$ is $\neg\neg$-separated if it satisfies

$$\forall x, y : X(\neg\neg[x \sim y] \to [x \sim y])$$

That is, there is a computable function φ such that for any $n \in [x \sim x], m \in [y \sim y]$, given that $[x \sim y]$ is non-empty, $\varphi \cdot \langle n, m\rangle$ terminates and $\varphi \cdot \langle n, m\rangle \in [x \sim y]$.

Each $\neg\neg$-separated object has a simpler description:

Definition 4.16. An *assembly* is a pair (X, E) where X is an underlying *carrier set*, and E is the *realizability relation*, a function $X \to \mathcal{P}(\omega)$, s.t. $E(x)$ is non-empty for every x.

Definition 4.17. A *morphism* of assemblies (X, E_X) and (Y, Y_E) is a function $f : X \to Y$, such that there is a recursive function $\{n\}$ that *tracks (or realizes)* f:

$$a \in E_X(x) \implies \{n\}(a) \in E_Y(f(x))$$

One can easily find computable functions tracking identity maps and composites; thus assemblies can be assembled together into a category Asm.

When unambiguous, we denote an assembly by its carrier set X and its realizability relation by E_X. Every assembly (X, E_X) can be viewed as an object of the effective topos, by taking the relation $\sim_X$ to be

$$\begin{cases} [x \sim_X x] = E_X(x) & \\ [x \sim_X y] = \emptyset & \text{if } x \neq y \end{cases}$$

Clearly, (X, E_X) is $\neg\neg$-separated, as witnessed by e.g. $\lambda x.\mathsf{p}_1 x$. The converse holds as well.

Proposition 4.18. *Every $\neg\neg$-separated object is isomorphic to an assembly.*

For this we would like to use the following auxiliary lemma.

Lemma 4.19. *Every object $(X, \sim)$ of $\mathcal{E}ff$ is isomorphic to an object $(X', \sim)$ such that $E_{X'}(x)$ is non-empty for every $x \in X'$.*

Proof. Put $X' = \{x \in X \mid E_X(x) \neq \emptyset\}$. The realizability relation on X' is then a restriction of $\sim$ to X'.

The morphism $F : (X', \sim) \to (X, \sim)$ is just the inclusion. The morphism in the other direction is $G : (X, \sim) \to (X', \sim)$ defined as $G(x, y) = [x \sim y]$. We leave it to the reader to verify that $F \circ G \simeq \mathrm{id}_X$ and $G \circ F \simeq \mathrm{id}_{X'}$.

□

Proof of proposition 4.18. If $(X, \sim)$ is $\neg\neg$-separated, then $(X, \sim)$ is isomorphic to the assembly $(X/R, E)$ where

- $R = \{(x, y) \mid [x \sim y] \neq \emptyset\}$
- $E([x]) = \bigcup_{x' \in [x]} [x' \sim x']$

Without loss of generality, we may suppose that all elements of X "exist" (i.e. $[x \sim x]$ is non-empty for every x). Then we define $f : X \to X/R$ by $f(x) = [x] = \{y \mid [x \sim y] \neq \emptyset\}$. Then if $a \in E_X(x) \implies a \in E_{X/R}([x])$ and $a \in [x \sim x'] \implies \mathsf{tr}(a, \mathsf{s}(a)) \in E_{X/R}([x]) = E_{x/R}([x'])$. This function f induces a morphism $F : (X, \sim) \to (X/R, E)$ in $\mathcal{E}ff$.

The morphism going in the other direction $G : (X/R, E) \to (X, \sim)$ is given by

$$G([x], y) = \begin{cases} \emptyset & \text{if } [y \sim x] = \emptyset \\ [y \sim y] & \text{otherwise} \end{cases}$$

We can see that G is a well-defined morphism:

(ST) If $a \in G([x], y)$, then $[x \sim y]$ is non-empty and thus $a \in E_{X/R}([x])$ and $a \in [y \sim y]$.

(REL) If $a \in G([x], y)$ and $b \in E([x])$ and $c \in [y \sim y']$, then $\mathsf{p}_1\langle b, a\rangle \in G([x], y)$ and $\mathsf{tr}(\mathsf{s}(c), c) \in G([x], y')$.

(TL) If $a \in E([x])$, then $a \in [x' \sim x']$ for some x' s.t. $[x \sim x']$ is non-empty; thus $a \in G([x], x')$.

(SV) Let $a \in G([x], y) = [y \sim y]$ and $b \in G([x], y') = [y' \sim y']$, then, because X is $\neg\neg$-separated, there is a recursive function ϕ such that $\phi ab \in [y \sim y']$ if $[y \sim y']$ is non-empty. However, because $a \in G([x], y)$ implies that $[x \sim y]$ is non-empty, and $b \in G([x], y')$ implies $[x \sim y']$ is non-empty, we know that $[y \sim y']$ is non-empty.

□

Thus, assemblies are exactly $\neg\neg$-separated objects in $\mathcal{E}ff$.

Discrete and modest sets. Another important subcategory of $\mathcal{E}ff$ is a subcategory $Mod \hookrightarrow Asm \hookrightarrow \mathcal{E}ff$ of *modest sets*. A modest set is a special case of a discrete object.

Definition 4.20. A *discrete object* is a quotient of a subobject of $\mathbf{N}$.

This definition may seem like a mouthful at first; we will therefore stick to the following characterization of discrete objects.

Proposition 4.21 ([29, Proposition 3.2.18])**.** *An object $(X', \sim')$ is* discrete *iff it is isomorphic to an object $(X, \sim)$ such that $n \in [x \sim x] \cap [y \sim y]$ implies $x = y$.*

From this point we will assume that all the discrete objects are given in the "canonical form" above.

Definition 4.22. A *modest set* is a discrete assembly.

Example 4.23. A natural numbers object $\mathbf{N}$ is a modest set.

It has been show by Van Oosten [28] that discrete objects can be intuitively understood as "discrete spaces". We will explain this analogy in proposition 4.33.

The notion of a discrete object can also be generalized to an arbitrary slice.

Definition 4.24. A map $f : (Y, \sim) \to (X, \sim)$ is discrete if it is a quotient of a subobject of the natural numbers object in the slice topos $\mathcal{E}ff/(X, \sim)$.

Once again, we would prefer to use a characterization of discrete maps in more concrete terms.

Proposition 4.25. *If a map $F : (Y, \sim) \to (X, \sim)$ is discrete, then the following proposition holds in realizability logic:*

$$F(y, x) \wedge F(y', x) \wedge [[y \sim y] \cap [y' \sim y']] \to [y \sim y']$$

In other words, every fiber $F^{-1}(x)$ is discrete.

The converse of the proposition above holds for assemblies; see [29, Proposition 3.4.4].

The full subcategory $\mathcal{D} \hookrightarrow \mathcal{E}ff$ of discrete objects is reflective ([29, Proposition 3.2.19]), and we will describe this reflection in geometrical terms in section 4.8.

Uniform objects.

Definition 4.26. An object $(X, \sim)$ is *uniform* if it is covered by an object in the image of ∇, that is, if there is an epimorphism $\nabla Y \to (X, \sim)$.

We would also make use of the following characterization:

Definition 4.27. [29, Proposition 2.4.7] An object is uniform if it is isomorphic to an object $(X, \sim)$ for which there is a $\psi \in \bigcap_{x \in X}[x \sim x]$.

Example 4.28. As a somewhat trivial example, any object $\nabla(X)$ is uniform with ψ being, e.g. 0.

Example 4.29. The canonical example of a uniform object is a powerset $\mathcal{P}(\mathbf{N})$. To see this, we first note that the realizability relation on $\mathcal{P}(\mathbf{N})$ can be rewritten as

$$P \Leftrightarrow Q := [\forall n : \mathbf{N}.(P(n) \to \{n\}) \wedge \forall n : \mathbf{N}.(P(n) \leftrightarrow Q(n))]$$

just by unfolding the definitions. We can "bundle up" the strictness of P in the definition of $\mathcal{P}(\mathbf{N})$, that is, we can consider predicates $P : \mathbf{N} \to \mathcal{P}(\omega)$, such that for all $x \in P(n)$, $\mathsf{p}_1 x = n$. Then $\mathcal{P}(\mathbf{N})$ is isomorphic to an object $(X, \sim)$, where X contains only such predicates $\mathbf{N} \to \mathcal{P}(\omega)$ that satisfies the aforementioned property, and

$$P \sim Q := [\forall n : \mathbf{N}.(P(n) \leftrightarrow Q(n))]$$

Then we can see that $(X, \sim)$ is uniform, and $\lambda x.x \in \bigcap_{P \in X}[P \sim P]$. In general, every powerset object is uniform, see [29, Proposition 3.2.6].

A small digression. The name *uniform* comes from the *uniformity principle* for the realizability interpretation of second-order arithmetic: $\forall X \exists x.\phi(X, x) \to \exists x \forall X.\phi(X, x)$, which is non-classical. In fact, the following uniformity principle holds in the topos logic of $\mathcal{E}ff$ for all formulas ϕ and for all uniform $(X, \sim)$ and discrete $(Y, \approx)$:

$$UP := \forall x : X \exists y : Y.\phi(x, y) \to \exists y : Y \forall x : X.\phi(x, y)$$

Note that for a uniform object $(X, \sim)$, the following statement are equivalent in realizability logic:

$$\begin{cases} [\forall x : X.([x \sim x] \to \phi(x, \bar{y}))] \leftrightarrow [\forall x : X.\phi(x, \bar{y})] \\ [\exists x : X.([x \sim x] \wedge \phi(x, \bar{y}))] \leftrightarrow \exists x : X.(\phi(x, \bar{y})) \end{cases}$$

Hence UP can be translated into realizability logic in the following way:

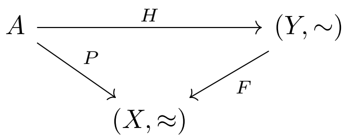
$$\forall x : X.\exists y : Y.([y \approx y] \wedge \phi(x, y)) \to \exists y : Y.([y \approx y] \wedge \forall x : X.\phi(x, y))$$

To see that UP is realized, suppose that n realizes the antecedent. Thus, $\mathsf{p}_1(n)$ gives us the realizer that is shared between all the witnesses for the existential clause for every $x : X$. Generally, a witness for x might be different than a witness for $x' \neq x$, but since $(Y, \approx)$ is discrete, different elements cannot share a realizer. We can conclude that there is a single $y \in Y$ that serves as a witness for every $x \in X$ in the antecedent of UP. It then follows straightforwardly that n realizes the consequent.

Uniform maps. A generalization of the previous notion is that of a *uniform map*.

Definition 4.30. A map $F : (Y, \sim) \to (X, \approx)$ is *uniform* if there is a set Z, an epimorphism $Q : \nabla(Z) \to \nabla\Gamma(X, \approx)$, and an epimorphism $H : A \to (Y, \sim)$ over $(X, \approx)$, where A is a pullback

$$\begin{array}{ccc} A & \longrightarrow & \nabla(Z) \\ \big\downarrow{\scriptstyle P} & & \big\downarrow{\scriptstyle Q} \\ (X, \approx) & \xrightarrow{\eta} & \nabla\Gamma(X, \approx) \end{array}$$



We will not go into the details on the motivation and theory behind uniform maps, but we will just mention that they provide a generalization of the uniformity principle to slices of $\mathcal{E}ff$, specifically uniform epimorphisms are (strong) orthogonal to discrete maps [29, Theorem 3.4.9].

The following characterization of uniform maps will also come in handy.

Proposition 4.31 ([29, Proposition 3.4.6])**.** *A map $F : (Y, \sim) \to (X, \approx)$ is uniform iff there are recursive functions α, β such that for all $y \in Y$, $x \in X$, $n \in [x \approx x]$, $m \in F(y, x)$ there exists an $y' \in Y$ and*

$$\begin{cases} \alpha(n) \in F(y', x) \\ \beta(n, m) \in [y \sim y'] \end{cases}$$

4.8 Van Oosten's path object construction

In this section we recall Van Oosten's *path object construction* [28] in $\mathcal{E}ff$, in which for every object X we associate an object $\mathsf{P}X$ of "paths" in X, making $\mathsf{P}X$ into an internal groupoid. Van Oosten uses a slightly different notion of a path than the one employed in this thesis, in particular the path object functor $X \mapsto \mathsf{P}X$ is not an exponent. The main motivation for Van Oosten's work is to find a way to understand the effective topos in topological term. Specifically, Van Oosten's path object is constructed in a such a way, that the terminology behind *discrete objects* and *discrete reflection* obtain an intuitive geometrical meaning. Specifically, a discrete object is an object containing only non-trivial Van Oosten's paths, and a discrete reflection of X is an object of path-connected components of X.

n-paths and discrete reflection. We start by defining the intervals in $\mathcal{E}ff$.

Definition 4.32. For each $n \in \mathbb{N}$ we define an n-interval I_n to be an assembly with the underlying set $\{0, \ldots, n\}$ and the realizability relation $E_{I_n}(i) = \{i, i+1\}$.

By n-path in an object $(X, \sim)$ we mean a map $I_n \to (X, \sim)$. As it was noted in [28], an interesting property of discrete objects is that they contain no non-trivial paths. In fact, we have the following proposition:

Proposition 4.33. *An object $(K, \sim)$ is discrete iff there are no non-constant n-paths $p : I_n \to K$, $n \geq 1$.*

Proof. Suppose that K doesn't have non-constant n-paths. Let $n \in [k_0 \sim k_0] \cap [k_1 \sim k_1]$ for $k_0, k_1 \in K$. Then consider a map $p : I_1 \to K$ induced by the set-level map $p(i) = k_i$ and tracked by $\lambda x.n$. Since this map is constant, we have to conclude $k_0 = k_1$.

For the other direction, suppose we have a map $P : I_n \to K$. By the totality of P, we have $\mathsf{tot}_P \in \bigcap_{i \in I_n}([i \sim i] \to \bigcup_{k_i \in K} P(i, k_i))$. Because $1 \in [0 \sim 0] \cap [1 \sim 1]$, we have

$$t := \mathsf{tot}_P \cdot 1 \in P(0, k_0) \cap P(1, k_1)$$

for some $k_0, k_1 \in K$. By the stability of P, we have $\mathsf{st}_P(t) \in [k_0 \sim k_0] \cap [k_1 \sim k_1]$. Hence, since K is discrete, $k_0 = k_1$. Similarly, $\mathsf{st}_P(\mathsf{tot}_P \cdot i) \in [k_{i-1} \sim k_{i-1}] \cap [k_i \sim k_i]$, and by the same argument we have that all k_i's are equal. We can now show that P is isomorphic to the constant map $r_{k_0} : I_n \to K$, $r_{k_0}(i, y) = [y \sim k_0]$. Let $n \in P(i, y)$. Then, by single-valuedness, $\mathsf{sv}_P(n, t) \in [y \sim k_i] = [y \sim k_0] = r_{k_0}(i, y)$. Thus, P is a constant path in K. □

It was also shown in [28] that discrete reflection of X can be seen as the collection of n-path-connected components of X. For an assembly X the construction can be replicated as follows: the discrete reflection X_d of X is obtained by quotienting X by the equivalence relation $\sim_P$ defined as

$$x \sim_P y \text{ if there is an } n\text{-path } p : I_n \to X, \text{ with } p(0) = x \text{ and } p'(m) = y$$

Thus $X_d = \{[x] \mid x \in X\}$ and $y \in [x]$ iff there is a sequence of realizers $a_0, \ldots, a_{n-1}$ and elements $x_1, \ldots, x_{n-1}$ of X such that $a_0 \in E(x) \cap E(x_1), a_1 \in E(x_1) \cap E(x_2), \ldots, a_{n-1} \in E(x_{n-1}) \cap E(y)$. Intuitively, a class $[x] \in X_d$ contains all points of X that are n-path-connected to x. The realizability relation on X_d is defined as

$$a \Vdash_{X_d} [x] \iff a \Vdash_X y \text{ for some } y \in [x]$$

The map $[–] : X \to X_d$ is tracked by $\lambda x.x$.

Proposition 4.34. *The construction $X \mapsto X_d$ restricts to a functor $Asm \to Mod$, which makes Mod a reflexive subcategory of Asm.*

Proof. Given a morphism $f : X \to Y$ (tracked by $\underline{f}$), we obtain a morphism $f_d : X_d \to Y_d$ defined as

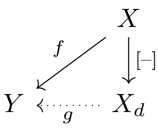
$$f_d([x]) = [f(x)]$$

and realized by $\underline{f}$. One can check that $(-)_d$ is indeed a functor.

To establish the reflection we have to verify two things: that X_d is indeed discrete, and that the arrow $[–]$ is universal.

For the first part, suppose that $[x], [y] \in X_d$, $[x] \neq [y]$. Then, by definition, $x \notin [y]$, and there is no path between x and y in X; in particular, there is no 1-path $p' : I_1 \to X$, such that $sp' = x$ and $tp' = y$. Thus, X_d is modest.

The second part amounts to filling in a dotted morphism g in the following diagram, where Y is a modest set:



We may simply put $g([x]) = f(x)$. To see that this is well-defined, suppose that $y \in [x]$, i.e. there is an m-path $p' : I_m \to X$ connecting x and y. Then there is a $a_0 \in E_X(x) \cap E_X(p'(1))$. Given a realizer $\underline{f}$ for f, we have $\underline{f} \cdot a_0 \Vdash_Y f(x), f(p'(1))$. Because Y is modest, $f(x) = f(p'(1))$. We can then apply this reasoning to $a_1 \in E_X(p'(1)) \cap E_X(p'(2))$ to get $f(p'(1)) = f(p'(2))$, etc. By induction we obtain $f(x) = f(y)$.

Finally, if $a \Vdash_{X_d} [x]$, then $a \Vdash_X y$ for some $y \in [x]$, and $\underline{f} \cdot a \Vdash_Y f(y) = f(x) = g([x])$. □

Path object. All the n-paths in X can be organized into a single object (path object); for that, some paths are identified via an *order and endpoint preserving map.*

Definition 4.35. A map $\sigma : I_n \to I_m$ is order and endpoint preserving iff it is order preserving and satisfies $\sigma(0) = 0$ and $\sigma(n) = m$.

We can now define a path object $\mathsf{P}(X, \sim)$ for a given object $(X, \sim)$ in $\mathcal{E}ff$.

Definition 4.36 (Van Oosten's path object)**.** The underlying set of $\mathsf{P}(X, \sim)$ is the set of all pairs (n, f) with $n \geq 1$ and f being a morphism $f : I_n \to (X, \sim)$. The realizability relation is defined as follows. For $(n, f), (m, g) \in \mathsf{P}(X, \sim)$ we have $\langle a, s, b \rangle \in [(n, f) \sim (m, g)]$ if

- $a \in E_{(X,\sim)^{I_n}}(f)$
- $b \in E_{(X,\sim)^{I_m}}(g)$

- Either there is an order and endpoint preserving function $\sigma : I_n \to I_m$ such that $s \in [f \sim g\sigma]$ in $(X, \sim)^{I_n}$; or, there is an order and endpoint preserving function $\sigma : I_m \to I_n$ such that $s \in [g \sim f\sigma]$ in $(X, \sim)^{I_m}$.

Proposition 4.37 ([28, Proposition 2.6]). *1. The construction of $\mathsf{P}(X, \sim)$ extends to an endofunctor $\mathsf{P} : \mathcal{E}ff \to \mathcal{E}ff$, which preserves finite limits.*

2. The object $\mathsf{P}(X, \sim)$ comes with well-defined maps

(a) $s, t : \mathsf{P}(X, \sim) \to (X, \sim)$ (source and target maps)

(b) $c : (X, \sim) \to \mathsf{P}(X, \sim)$ (constant map)

(c) $* : \mathsf{P}(X, \sim) \times_{(X,\sim)} \mathsf{P}(X, \sim) \to \mathsf{P}(X, \sim)$ (composition of paths)

(d) $\tilde{\cdot} : \mathsf{P}(X, \sim) \to \mathsf{P}(X, \sim)$ (path reversal)

Proposition 4.38 ([28, Proposition 2.7]). *There is a a morphism $L : \mathsf{P}(X, \sim) \to \mathsf{PP}(X, \sim)$ "contracting a path onto its endpoint", i.e. satisfying internally $s(L(p)) = p$, $t(L(p)) = c(t(p))$.*

The two propositions above imply that $\mathcal{E}ff$ possesses the structure of a (nice) path object category [6, 12], which in turn gives rise to a model of type theory in which the identity type of X is interpreted as $\mathsf{P}X$. We will compare the model with ours in the upcoming chapter.

Notes

In this chapter we recalled the definition of the effective topos, some of its properties and various classes of objects from the theory of realizability toposes. The topic of categorical realizability is vast and this purpose of this chapter is to make the thesis self-contained. We can recommend lecture notes [38, 3] to a reader interested in basic categorical realizability. The book [29] is a grand reference for realizability triposes and toposes.

In the next chapter we apply the construction of chapter 3 to the effective topos. We also discuss the differences and similarities to the approach of Van Oosten.

Chapter 5

Model category structure on $\mathcal{E}ff_f$

This chapter is devoted to the study of the model category structure on $\mathcal{E}ff$, as presented in chapter 3. First, we describe the interval object, given by $\nabla 2 \simeq I_1$. The we study the relation between the notions of fibrant and contractible objects (types) on the one hand, and notions of uniform maps and discrete maps in $\mathcal{E}ff$ on the other hand. We show that in Asm, contractible maps/trivial fibrations are exactly uniform maps. We also show that for fibrant objects the notions of uniformity and contractibility coincide. Next, we show that discrete objects are fibrant, and that discrete maps with discrete bases are fibrations. We discuss importance of the discrete reflection and use it to show that the homotopy category of fibrant assemblies is equivalent to the category of modest sets. Finally, we contrast our approach with Van Oosten's notion of homotopy based on the path object construction.

5.1 Intervals in $\mathcal{E}ff$

The interval. We take interval object I to be $I := \nabla(2)$. The endpoint inclusions $1 \simeq \nabla(1) \to \nabla(2) = I$ are given by $* \mapsto 0$ and $* \mapsto 1$.

The interval object I supports connections via maps $I \times I \xrightarrow{c_0} I$ and $I \times I \xrightarrow{c_1} I$:

$$\begin{aligned} c_0(x,y) &= min(x,y) \qquad \text{tracked by } \lambda x.0 \\ c_1(x,y) &= max(x,y) \qquad \text{tracked by } \lambda x.0 \end{aligned}$$

Our interval object is in fact isomorphic to I_1, allowing us to employ proposition 4.33.

Proposition 5.1. *I is isomorphic to I_1.*

Proof. The isomorphism is the identity function, which is tracked by $\lambda x.1$ in both directions. □

Recall that one can view objects I_n as intervals of "length" n. In fact, all intervals I_{n+1} are obtained by gluing together n copies of I. For $I_{n+1} = I_n +_1 I$:

$$\begin{array}{ccc} 1 & \xhookrightarrow{0 \mapsto 0} & I_1 \\ {\scriptstyle 0\mapsto n}\big\downarrow & & \big\downarrow \\ I_n & \longrightarrow & I_n +_1 I_1 \end{array}$$

It is clear that on the level of sets, $|I_{n+1}| \simeq |I_n +_1 I_1|$, since the pushout construction identifies $(0, n)$ and $(1, 0)$.

The isomorphism is tracked by $\lambda\langle i,j\rangle.(i\times n)+j$ in one direction, and is tracked by

$$\lambda x.\begin{cases} x \le n & \to \langle 0, x\rangle \\ x > n & \to \langle 1, x-n\rangle \end{cases}$$

in the other direction. Thus, an interval I_n can be seen as a chain of 1-intervals and an n-path can be seen as a chain of 1-paths.

Paths in the objects of $\mathcal{E}ff$. Suppose we have an assembly (X, E_X). Since the interval I is also an assembly, a *path* $p : I \to X$ is a completely determined by its endpoints $p(0)$ and $p(1)$ that share a realizer $n \Vdash_X p(0), p(1)$. Conversely, if $x, y \in X$ share a realizer n, then the function $p(i) = \begin{cases} x & \text{if } i = 0 \\ y & \text{if } i = 1 \end{cases}$ is tracked by $\lambda x.m$. A similar result can be obtained for an arbitrary object.

Proposition 5.2. *Any map $F : I \to (X, \sim)$ is induced by a set-level map.*

Proof. By the totality of F, we have $\mathsf{tot}_F \cdot 1 \in \bigcup_{x:X} F(i,x)$ for every $i \in I$. That is, $\mathsf{tot}_F \cdot 1 \in F(0,x_0) \cap F(1,x_1)$ for some $x_0, x_1 \in X$. Then, by stability of F, we have $\kappa := \mathsf{st}_X(\mathsf{tot}_F \cdot 1) \in [x_0 \sim x_0] \cap [x_1 \sim x_1]$.

We are going to show that F is induced by a set-level function $f : i \mapsto x_i$ which is tracked by $\lambda m.\kappa$. By a straightforward calculation, f induces a morphism F':

$$F'(i,x) = E_I(i) \wedge [f(i) \sim x] \simeq [x \sim x_i]$$

We are to show that $F' \simeq F$. Suppose we have $n \in F'(i,x) = [x \sim x_i]$. Then, by totality of F we get $\mathsf{tot}_F \cdot 1 \in F(i, x_i)$ (uniformly in i). Because F respects $\sim$, we have $\mathsf{rel}_F(1, \mathsf{s}(n), \mathsf{tot}_F \cdot 1) \in F(i,x)$. ☐

This implies that we can describe the "path space" X^I as an object $(X^I, \approx)$ of maps $f : I \to X$, where

$$[f \approx g] := [tr(f) \wedge \forall i : I([f(x) \sim g(x)])]$$

where $tr(f)$ states that f is tracked by some recursive function.

5.2 Fibrant and contractible objects

By the results in chapter 3, the interval object I induces a model structure on the subcategory $\mathcal{E}ff_f$ of fibrant objects of the effective topos. In this section we wish to study fibrant and contractible objects of $\mathcal{E}ff_f$ w.r.t. that model structure.

5.2.1 Uniform and contractible objects

Below we describe some homotopical properties of uniform objects and uniform maps.

Uniform objects. Recall that an object $(Y, \sim)$ is said to be *uniform*, if it is covered by an object in the image of $\nabla : \mathbf{Set} \to \mathcal{E}ff$. That is, there is a set X and an epimorphism $Q : \nabla X \to (Y, \sim)$. A non-categorical description of uniform objects is the following: an object $(Y, \sim)$ is uniform if there is $n \in \bigcap_{y \in Y}[y \sim y]$.

As it turns out, there is a connection between contractible (trivially fibrant objects) and uniform objects, which we wish to explore in this section.

Proposition 5.3. *If a uniform object $(X, \sim)$ has a global element $s : 1 \to X$, then X is homotopic to 1.*

Proof. Clearly, $!_X \circ s = \mathrm{id}_1$, so it suffices to establish a homotopy $s \circ !_X \sim \mathrm{id}_X$. But first we replace s with an equivalent map $s' : 1 \to X$. By the totality of s, we have some $n \in \bigcup_{x:X} s(*, x)$, i.e. $n \in s(*, c)$ for some $c \in X$. Then we put $s'(*, x) = [x \sim c]$. One can see that s' is a well-defined morphism. Furthermore, given $s(*, x)$, we can get $s'(*, x) = [x \sim c]$ from the strictness and totality of s. Hence, $s' = s$.

Then consider a homotopy $\theta : I \times X \to X$ defined as

$$\begin{cases} \theta(0, x, y) = s'(*, y) = [y \sim c] \\ \theta(1, x, y) = [x \sim y] \end{cases}$$

Clearly, θ is strict and single-valued. To see that θ is total, it suffices to provide an element $\theta(0, x, y_0) \cap \theta(1, x, y_1) = [y_0 \sim c] \cap [x \sim y_1]$ for some y_0, y_1 given $[x \sim x]$. But if we take $y_0 = c$ and $y_1 = x$, then the required element can be obtained from uniformity of X. ☐

It is worth noting that in the proposition above, we make use of the fact that the notion of homotopy equivalence is defined for all objects. However, the notion of homotopy corresponds to geometrical intuition only on fibrant objects, for which we have an immediate corollary.

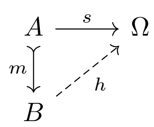
Corollary 5.4. *If a uniform object is fibrant and inhabited, then it is contractible.*

Proof. Follows from the proposition above, using theorem 3.8. ☐

In our model cofibrations are exactly monomorphisms. Thus, trivially fibrant/contractible objects are those that have the right lifting property against monomorphisms. In category theory such objects are called *injective*.

Example 5.5. The subobject classifier Ω is contractible. For any X, the powerobject Ω^X is contractible.

Proof. This follows from the fact that in a topos a subobject classifier is *injective* (see e.g. [24, Corollary 1.27]). Given a lifting problem



We can define a filler $h : B \to \Omega$ "set-theoretically" as $h(b) = \bigvee_{a \in m^{-1}(b)} s(a)$.

We can also see that Ω is uniform, as $\langle \mathrm{id}, \mathrm{id} \rangle \Vdash \bigcap_{A:\Omega} [A \leftrightarrow A]$. ☐

The interval object I is uniform. It is natural to ask whether it is contractible. This amounts to asking if every map $k : A \to I = \nabla(2)$ can be extended along a monomorphism $m : A \to B$. The object I is $\neg\neg$-separated, hence a map into I is induced by a set-level map that is tracked by a recursive function. However, since I is uniform, the tracking function provides no additional information; for instance, every well-defined function into I is tracked by $\lambda x.1$. Thus, the question whether I is trivially fibrant can be restated in the following way: can every map $k : A \to 2$ of sets be extended along an injection $m : A \hookrightarrow B$? In other words, is the object $2 = \{0, 1\}$ injective in **Set**?

As it turns out, this statement depends on certain facts about the ambient set theory. As it shown in [1, Theorem 8], the statement that 2 is injective is equivalent to a weak form of the restricted law of excluded middle, that is $\neg\varphi \vee \neg\neg\varphi$ for a formula φ in which every quantifier is bounded by a set (such formula is said to be *restricted*).

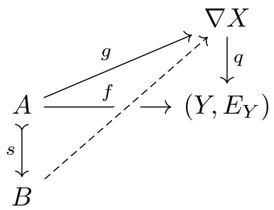
In fact, we can apply the same reasoning to any inhabited uniform assembly. Then the statement that every inhabited uniform assembly is fibrant (and, thus, contractible) is equivalent to the statement that every inhabited set is injective. From [1, Proposition 5], the statement that every inhabited set is injective in CZF is equivalent to the restricted law of excluded middle, that is $\varphi \vee \neg\varphi$ for restricted formulas.

Proposition 5.6. *Any uniform (non-empty) assembly* (Y, E_Y) *is injective (contractible), given that* **Set** *satisfies the law of excluded middle for restricted formulas and axiom of choice.*

Proof. Let (Y, E_Y) be a uniform assembly covered by $q : \nabla X \to (Y, E_Y)$. Suppose we have a monomorphism $s : A \rightarrowtail B$ and we are to extend a map $f : A \to (Y, E_Y)$ along it.

Using axiom of choice, for each $y \in Y$ such that $E_Y(y) \neq \emptyset$ we pick $x_y \in q^{-1}(y)$. Then we can define a function $g : A \to \nabla X$ by $g(a) = x_{f(a)}$. This map is trivially realized, e.g. by $\lambda x.0$. Furthermore, $q \circ g = f$.

Then, since ∇X is injective (this fact involves restricted LEM on the level on sets, as per discussion in the previous paragraphs), hence we can extend g along s and obtain a map $h : B \to \nabla X$ such that $h \circ s = g$. Then one can check that $q \circ h$ is the desired extension of f.



□

However, if we go beyond assemblies, the situation is a bit different. In general, trivially fibrant objects are uniform, but not the other way around. First of all, we have the following observation about injective objects.

Proposition 5.7. *If an object* $(X, \sim)$ *is injective (contractible), then it is uniform.*

Proof. This is a special case of the fact that in a topos, every injective object is a retract of a powerobject. Consider a monomorphism $\{\cdot\} : (X, \sim) \to \Omega^{(X,\sim)}$ which sends x to a singleton $\{x\}$. Because $(X, \sim)$ is injective, we can extend the identity map $\mathrm{id}_X : (X, \sim) \to (X, \sim)$ along the monomorphism $\{\cdot\}$ to obtain a retraction $R : \Omega^{(X,\sim)} \to (X, \sim)$.

By [29, Proposition 3.2.6], $\Omega^{(X,\sim)}$ is uniform, so it is covered by some $\nabla Z \to \Omega^{(X,\sim)}$; hence, $(X, \sim)$ is covered by ∇Z as well. □

The other direction does not hold, which we will illustrate with the following counterexample.

A "circle". As we have seen above, contractible objects can be characterized as fibrant and uniform objects. The fibrancy of a uniform object is crucial for this characterization, as the other direction of proposition 5.7, does not hold. For this we consider a "circle" S which we obtain by gluing together two endpoints of the interval using a coequalizer diagram:

$$1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} (I, \sim) \xrightarrow{q} (S, \approx)$$

Explicitly, using the formulation from section 4.4, $S = \{0, 1\}$ and $[i \approx j] = [\forall A : \mathcal{P}(\omega)^{I \times I}.(Equiv(A) \wedge A(0,1)) \to A(i,j)]$.

The circle is covered by $I = \nabla(2)$ so it is uniform. However, we will show that it is not contractible.

Consider two paths $P, Q : I \to S$

$$P(i,j) = [i \approx j] \qquad Q(i,j) = [0 \approx j]$$

Intuitively, P goes "around" the "circle" and Q just stays at 0. We can check that both paths have the same endpoints, yet $P \neq Q$

Proposition 5.8. $(P \circ \partial)(i,j) \simeq (Q \circ \partial)(i,j)$ *for* $i, j \in \{0, 1\}$*, but* $P \neq Q$*; where* $\partial = [\delta_0, \delta_1] : 2 \hookrightarrow I$.

Proof. This reduces to showing

$$\{i\} \wedge P(i,j) = \{i\} \wedge [i \approx j] \simeq \{i\} \wedge Q(i,j) = \{i\} \wedge [0 \approx j]$$

which can be done uniformly by inspecting the $\{i\}$.

To show that $P \neq Q$ assume the opposite. Then there is a computable $\varphi : [0 \approx j] \to [i \approx j]$ uniform in i, j. Then consider a projection $\pi \in [(Equiv(A) \wedge A(0,1)) \to A(0,1)] = [0 \approx 1]$. Then $\varphi \cdot \pi \in [i \approx 1]$. In other words, $\varphi \cdot \pi \in [0 \approx 1] \cap [1 \approx 1]$.

Consider an equivalence relation A defined as

$$\begin{cases} A(i,i) = \{\langle 1 \rangle, \langle i, i \rangle\} \\ A(i,j) = \{\langle i, j \rangle\} & \text{if } i \neq j \end{cases}$$

One can check that A is indeed an equivalence relation. Let $n \in Equiv(A)$. Then $\varphi \cdot \pi \cdot \langle n, \langle 0, 1 \rangle\rangle \in A(0,1) \cap A(1,1)$, but $A(0,1) \cap A(1,1) = \emptyset$.

□

In fact, we have a stronger result.

Proposition 5.9. *There is no constant on the endpoints homotopy $P \sim_{S \times S} Q$.*

Proof. Suppose $H : I \times I \to S$ is such a homotopy. Then, by a straightforward calculation, we have

$$\begin{cases} H(0,i,k) \simeq P(i,k) = [i \approx k] \\ H(1,i,k) \simeq Q(i,k) = [0 \approx k] \end{cases}$$

Because H is constant on the endpoints, we have

$$\begin{cases} (H \circ (I \times \delta_0)) \simeq r_0 \\ (H \circ (I \times \delta_1)) \simeq r_1 \end{cases}$$

where $r_0 = P \circ \delta_0 \circ !_I = P \circ \delta_1 \circ !_I$. Specifically,

$$\begin{cases} (H \circ (I \times \delta_0))(i,k) = H(i,0,k) \simeq P(0,k) \simeq Q(0,k) \\ (H \circ (I \times \delta_1))(i,k) = H(i,1,k) \simeq P(1,k) \simeq Q(1,k) \end{cases}$$

uniformly in i, k. Hence we have $\psi \in [\forall i.(Q(1,k) \to H(i,1,k))]$. Using the same $\pi \in [0 \approx 1] = Q(1,1)$ from the previous proposition we have $\psi \cdot \pi \in H(i,1,1)$, i.e. $\psi \cdot \pi \in H(0,1,1) \cap H(1,1,1) = P(1,1) \cap Q(1,1) = [1 \approx 1] \cap [0 \approx 1]$. Using the same equivalence relation A and by the same argument as in the previous proposition, we obtain a contradiction.

□

Because of the previous proposition, we know that S is not an "h-set". Specifically, if S would be contractible, then by a standard argument ([40, Lemma 3.3.4]) it would be a homotopy-set: every two paths with the same endpoints would have been homotopical over $\langle s, t \rangle : S^I \to S \times S$, which is not the case as demonstrated in proposition 5.9. Together with proposition 5.3 this implies that S is not fibrant. Therefore, we can conclude that not every inhabited uniform object is contractible.

Uniform maps. Let us turn our attention to uniform maps. Recall from proposition 4.31, that a map $F : (Y, \sim) \to (X, \approx)$ is uniform iff there are recursive functions α, β such that for all $y \in Y$, $x \in X$, $n \in [x \approx x]$, $m \in F(y,x)$ there exists an $y' \in Y$ and

$$\begin{cases} \alpha(n) \in F(y', x) \\ \beta(n, m) \in [y \sim y'] \end{cases}$$

Suppose that both Y and X are assemblies. Then the condition for a morphism f above can be reduced to: for all $y \in Y, n \in E_X(f(y))$, $\langle m_1, m_2 \rangle \in E_X(f(y)) \wedge E_Y(y)$ there is $y' \in Y$ such that

$$\begin{cases} \alpha(n) \in E_X(f(y')) \wedge E_Y(y') \\ \beta(n,m) \in [y' \sim y] \text{ so } y' = y \end{cases}$$

This basically states that given $n \in E_X(x)$ such that $f^{-1}(x)$ is non-empty, we can find $\mathsf{p}_2(\alpha(n)) \in \bigcap_{y \in f^{-1}(x)} E_Y(y)$. Which is to say that every non-empty fiber of f is uniform. We say that $\lambda x.\mathsf{p}_2(\alpha(x))$ "witnesses" the uniformity of f.

As it turns out, for assemblies notions of a uniform epimorphism and a trivial fibration coincide.

Proposition 5.10. *A uniform epimorphism between assemblies is a trivial fibration, given the axiom of choice in* **Set**.

Proof. Suppose then we have a commutative square as follows, where f is an epi:

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{h} & X \end{array}$$

Then we can construct the filler k for this lifting problem in **Set** (that is, if we consider the square under the image of Γ), as f is epi and i is mono[1]. We can show that this function k is tracked. For this, notice that for any $b \in B$, $k(b) \in f^{-1}(h(b))$. But in the presence of a realizer for $h(b)$, we can find a uniform realizer for $f^{-1}(h(b))$, as in discussion above. Hence, k is tracked by $\lambda x.\mathsf{u}(\underline{h} \cdot x)$ where $\underline{h}$ tracks h and u witnesses the uniformity of f. □

A trivial fibration is always an epimorphism (in fact, it's a split epimorphism as it has a section).

Proposition 5.11. *A trivial fibration between assemblies is uniform.*

Proof. Suppose that $f : Y \to X$ is a trivial fibration tracked by $\underline{f}$. Then consider an assembly ΣY with the underlying set $\{(x,y) \in X \times Y \mid f(y) = x\}$ and realizability relation $E_{\Sigma Y}(x,y) = E_X(x)$.

Consider a map $f \times \mathrm{id} : Y \to \Sigma Y$ sending $y \in Y$ to $(f(y), y) \in \Sigma Y$. This map is tracked by $\underline{f}$ and is a monomorphism, as $(f \times \mathrm{id})(y) = (f \times \mathrm{id})(y') \implies y = y'$. Hence the following lifting problem has a solution h:

$$\begin{array}{ccc} Y & = & Y \\ {\scriptstyle f \times \mathrm{id}}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle f} \\ \Sigma Y & \xrightarrow[\pi_1]{} & X \end{array}$$

The map h is tracked by some recursive φ. One can check that φ satisfies

$$n \in E_X(x) \implies \varphi(n) \in \bigcap_{y \in f^{-1}(y)} E_Y(y)$$

□

This result states that, although in general case contractible objects are exactly those uniform objects that are fibrant, for assemblies the condition of fibrancy can be dropped.

To sum up, our state of knowledge about uniform and contractible maps can be depicted in the following table.

[1]Axiom of choice in **Set** is equivalent to the statement that (mono,epi) is a weak factorisation system in **Set**.

Uniform assembly	$\Longleftrightarrow$	Contractible assembly
Uniform epimorphism between assemblies	$\Longleftrightarrow$	Trivial fibration between assemblies
Uniform object	$\not\Longrightarrow$ $\Longleftarrow$	Contractible object
Fibrant uniform object	$\Longleftrightarrow$	Contractible object

5.2.2 Modest sets and discrete objects

Let us turn our attention to discrete objects and discrete maps.

Proposition 5.12. *Every discrete object X is fibrant.*

Proof. Suppose we are given a lifting problem

$$\begin{array}{ccc} (\{0\}\times B)\cup(I\times A) & \xrightarrow{[\alpha_0,\alpha_1]} & X \\ {\scriptstyle \delta_0\hat{\otimes}u}\big\downarrow & & \big\downarrow \\ I\times B & \xrightarrow{\qquad\quad} & 1 \end{array}$$

First of all, we note that $\alpha_1(0,a,x)\le\alpha_1(1,a,x)$. For given $[a\sim a]$ we have, by totality, $n\in\bigcup_{x:X}\alpha_1(i,a,x)$, i.e. $n\in\alpha_1(0,a,x_0)\cap\alpha_1(1,a,x_1)$. Then, $st_X(n)\in[x_0\sim_X x_0]\cap[x_1\sim_X x_1]$. Since X is discrete, $x_0=x_1$. That means that given $\alpha_1(0,a,x)$ we have $[x\sim_X x_0]$ and $\alpha_1(1,a,x_1)=\alpha_1(1,a,x_0)$; from this we can obtain $\alpha_1(1,a,x)$. In addition, given $\alpha_1(0,a,x)$ and $\alpha_1(0,a,x_0)\cap\alpha_1(1,a,x_0)$ we can obtain $\alpha_1(0,a,x)\cap\alpha_1(1,a,x)$ by strictness w.r.t. $[x\sim x_0]$.

Therefore, we can define a filler $H:I\times B\to X$ as $H(i,b,x)=\alpha_0(b,x)$. Clearly, $H\circ(\delta_0\times B)\simeq\alpha_0$. To see that $H\circ(I\times u)\simeq\alpha_1$ note

$$(H\circ(I\times u))(i,a,x)\simeq P_u(a)\wedge H(i,a,x)\simeq P_u(a)\wedge\alpha_0(a,x)$$

where P_u is the strict relation determining the subobject $u:A\rightarrowtail B$. Since α_0 and α_1 "agree" on $\{0\}\times A$, we may derive from the above equation $\alpha_1(0,a,x)$. Then, using the reasoning outlined at the beginning of this proof we can establish that $\alpha_1(i,a,x)$. □

The above proposition is a special case of the following fact.

Proposition 5.13. *Given a discrete family $p:M\to X$, if X is a discrete object, then p is a fibration.*

Proof. For simplicity, we only show this for modest sets. Given a lifting problem

$$\begin{array}{ccc} (\{0\}\times B)\cup(I\times A) & \xrightarrow{[\alpha_0,\alpha_1]} & M \\ {\scriptstyle \delta_0\hat{\otimes}u}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ I\times B & \xrightarrow[\beta]{\qquad\quad} & X \end{array}$$

we construct a lift $H:I\times B\to M$ as in the previous proposition: $H(i,b)=\alpha_0(b)$. Since X is modest, we have $\beta(0,b)=\beta(1,b)$ by the same reasoning as in the previous proposition. But $\beta(i,b)=\beta(0,b)=p(\alpha_0(b))=p(H(i,b))$. Thus, the lower triangle commutes.

As for the upper triangle, we have to verify that $\alpha_1(i,a)=H(i,u(a))$. But $H(i,u(a))=\alpha_0(u(a))=\alpha_1(0,a)$. Thus it suffices to verify that $\alpha_1(1,a)=\alpha_1(0,a)$. This holds because M itself is a modest set. For if there is a non-trivial path in M, it cannot lie in a single fiber (for each fiber is modest). Thus it has to lie over a non-trivial path in X – which is impossible since we assumed that X is modest itself. □

In general, a discrete family need not be a fibration, as witnessed by the following counterexample.

Example 5.14. $2 \xrightarrow{[\delta_0,\delta_1]} I$ is a modest family that is not a fibration.

Proof. The fiber 2_i is exactly $\{i\}$, and is modest. However, consider the following lifting problem:

$$\begin{array}{ccc} (\{0\} \times I) \cup (I \times \{0\}) & \xrightarrow{[\phi,\phi]} & 2 \\ {\scriptstyle \delta_0 \hat{\otimes} \delta_0} \downarrow & & \downarrow {\scriptstyle [\delta_0,\delta_1]} \\ I \times I & \xrightarrow{\quad c_1 \quad} & I \end{array}$$

where ϕ is a constant map that sends $i \in \{0, 1\}$ to $0 \in 2$. Suppose this diagram has a filler $h : I \times I \to 2$. Consider a "diagonal" $d : I \to I \times I$, $d(i) = (i, i)$. Then $c_1 \circ d = \mathrm{id}_I : I \to I$ is a non-trivial path in I; the composite $h \circ d : I \to 2$ would then be a non-trivial path in 2, which is a contradiction.

□

Example 5.15. A direct consequence of proposition 5.12 is that the natural numbers object $\mathbf{N}$ is fibrant.

Discrete reflection. Now we turn our attention to the *discrete reflection*. Recall that the category $\mathcal{D}$ of discrete objects is a reflexive subcategory of $\mathcal{E}ff$; likewise, the category *Mod* of modest sets is a reflexive subcategory of *Asm*.

As we have seen in section 4.8, the discrete reflection X_d of an assembly X can be described as a set of path-connected components of X, using Van Oosten's definition of an n-path. Specifically, the underlying set of X_d is $\{[x] \mid x \in X\}$, where x and y are in the same equivalence class if there is an n-path connecting the two points. In our model, if X is fibrant, then there is a composition operation on paths (by proposition 3.13); so, if $x, y \in X$ are connected by a chain of paths, then there is a single path $p : I \to X$ connecting the two. It follows that the equivalence class $[x] \in X_d$ can be defined as

$$[x] := \{y \in X \mid \exists p : I \to X.p(0) = x \wedge p(1) = y\}$$

This construction implies that a fibrant assembly is homotopic to its discrete reflection. In order to prove this, we need the following proposition.

Proposition 5.16 ([17, Proposition 2.12])**.** *The map $p : E \to X$ is a fibration if and only if $\hat{\exp}(\delta_0, p)$ and $\hat{\exp}(\delta_1, p)$ are trivial fibrations.*

Proof. Similar to the proof in section 2.1.1, using the fact that there is one-to-one correspondence between commutative squares $\delta_e \hat{\otimes} u_i \Rightarrow p$ and $u_i \Rightarrow \hat{\exp}(\delta_e, p)$, with $u_i \in \mathcal{I}$ and $e \in \{0, 1\}$. □

In particular, the map $!_X : X \to 1$ is a fibration if an only if maps $\hat{\exp}(\delta_0, !_x) = s : X^I \to X$, $\hat{\exp}(\delta_1, !_x) = t : X^I \to X$ are trivial fibrations. Equipped with this fact we can prove that the reflection map is a homotopy equivalence.

Proposition 5.17. *Given a fibrant assembly X, the reflection map $[-] : X \to X_d$ is a homotopy equivalence, assuming the axiom of choice in the ambient set theory.*

Proof. Using the axiom of choice, we can pick for each equivalence class $[x]$ a representative $g([x]) \in [x]$. Then, for each element $y \in [x]$, there is a path $p_y : I \to X$ with $p_y(0) = y$ and $p_y(1) = g([x])$. We shall see that the map g induces a morphism $g : X_d \to X$.

Because X is fibrant, by proposition 5.16, $s : X^I \to X$ is a trivial fibration. By proposition 5.11, s is a uniform map; by characterization of uniform maps, there is a recursive function u such that $\mathsf{u} \cdot n \in \bigcap\{E_{X^I}(p) \mid p \in s^{-1}(x)\}$ for $n \in E_X(x)$. We have a chain of implications

$$n \Vdash_{X_d} [x] \iff n \Vdash_X y \text{ for some } y \in [x] \implies \mathsf{u} \cdot n \Vdash_{X^I} p_y$$

Hence $\mathsf{u} \cdot n \cdot 1 \Vdash_X g([x]), y$; and g is tracked by $\lambda x.\mathsf{u} \cdot x \cdot 1$.

It is clear, that g is a section of $[-]$; it remains to show that $g([-]) \sim \mathrm{id}_X$. Intuitively, this holds because $g([x]) \in [x]$ and thus connected to x by a path. We can further show that this choice of path is uniform/continuous in x. We have a homotopy $\Theta : I \times X \to X$, given by

$$\begin{cases} \Theta(0, x) = & x \\ \Theta(1, x) = & g([x]) \end{cases}$$

One can check that it is tracked by $\lambda\langle i, x\rangle.\mathsf{u} \cdot x \cdot 1$. □

This fact implies that in the homotopy category $Ho(\mathcal{E}ff_f)$, assemblies are identified with modest sets. Thus, the subcategory $Ho(Asm_f) \hookrightarrow Ho(\mathcal{E}ff_f)$ is equivalent to the homotopy category $Ho(Mod_f)$ of modest sets. However, because modest sets have no non-trivial paths, there are no non-trivial homotopies in Mod – in other words, if two modest sets are homotopy equivalent, they are isomorphic; thus, inverting homotopy equivalences will not add additional isomorphisms to Mod, and Mod_f is equivalent to $Ho(Mod_f)$. We can concisely state the conclusion of this discussion as

Proposition 5.18. $Ho(Asm_f) \simeq Mod$

5.3 Comparison with Van Oosten's notion of homotopy

In this section we contrast our model with that of Van Oosten, which was described in section 4.8. We give a counterexample for functional extensionality for that model. The chapter is concluded by showing that while our model has "fewer types" (not all objects are fibrant), for fibrant assemblies our construction of a path object is homotopic to that of Van Oosten.

5.3.1 Functional extensionality

By "functional extensionality" we mean a map $e : \mathsf{P}(X)^Y \to \mathsf{P}(X^Y)$, such that

$$(\langle s, t\rangle)^Y = \langle s, t\rangle \circ e$$

Van Oosten's path object construction does not support functional extensionality, as can be witnessed by the following example:

Example 5.19 ([27])**.** Consider an assembly L defined as a set $\{0, 1, \dots\}$ with the realizability relation $E_L(x) = \{x, x + 1\}$ and functions $f, g : L \to L$ defined as

$$f(x) = 0 \qquad g(x) = x$$

with obvious realizers. The two functions are homotopical: for each $x \in L$, there is a path (in the sense of Van Oosten) between 0 and x, the homotopy $\theta : L \to PL$ is defined as $\theta(x) = [(x, \mathrm{id})]$, and tracked by $\lambda xy.y$.

However, there is no path (in the sense of Van Oosten) between f and g in L^L. For assume there is one; then it is represented by a tuple $(n, p : I_n \to L^L)$, s.t. $\langle s, t\rangle \circ p = (f, g)$. Furthermore, p is tracked by φ.

Hence, $\varphi \cdot 0$ has to track f, and thus $\varphi \cdot 0$ can only take values in $\{0, 1\}$, as $rng(f) = \{0\}$. It is also the case that $\varphi \cdot 1$ tracks both f and $p(1)$,thus $\varphi \cdot 1$ as well takes values in $\{0, 1\}$, as $\varphi \cdot 1$ tracks f. Since it also tracks $p(1)$ we can deduce that $rng(p(1)) \subseteq \{0, 1\}$. Similarly, we can show that $rng(p(2)) \subseteq \{0, 1, 2\}$, and so on. One can show, by induction, that $rng(p(n)) = rng(g) \subseteq \{0, \dots, n\}$, which is clearly not the case.

5.3.2 Homotopy between the path objects

Intuitively, the Van Oosten's path object $\mathsf{P}(X)$ is "richer" then the path space X^I, as it contains paths of inevitably variable length. However, in case when X is a fibrant assembly, there is a homotopy equivalence $\mathsf{P}(X) \sim X^I$. In order to show this it suffices to provide a path between $[(n,q)]$ and $[(1,p)]$ in $\mathsf{P}(X)$ with $p(0) = q(0)$ and $p(1) = q(n)$. For this we use the fibrancy of X to fill "triangles" in X containing $p(0), q(i)$ and $q(i+1)$ as vertices.

Theorem 5.20. *If X is a fibrant assembly, then there is a recursive function $\underline{tr}$ such that if $\alpha \Vdash q : I_n \to X$, then $\underline{tr}\langle \alpha, n\rangle \Vdash q(0), q(n)$.*

Proof. Let us write $\underline{c}$ for a recursive function realizing the composition operation c of proposition 3.13. Suppose $q : I_n \to X$ is an n-path in X and $\alpha \Vdash q$. But then $\alpha \Vdash q_0 : I \to X$ where $q_0(i) = q(i)$. Similarly, $\lambda x.\alpha \cdot (x+1) \Vdash q_1 : I \to X$ where $q_1(i) = q(i+1)$. Thus $\underline{c}\langle \alpha, \lambda x.\alpha \cdot (x+1)\rangle \Vdash p_0 : I \to X$ where $p_0(0) = q_0(0), p_0(1) = q_1(1)$. We can iterate this process. Define

$$\mathsf{pad}(\alpha, i) = \lambda x.\alpha \cdot (x+i)$$

Then we know

$$\underline{c}\langle \underline{c}\langle \alpha, \mathsf{pad}(\alpha,1)\rangle, \mathsf{pad}(\alpha,2)\rangle$$

tracks $p_2 : I \to X$ defined as $p_2(0) = q(0), p_2(1) = q(2)$. We can implement such composition uniformly for any n, using primitive recursion. Thus we obtain a function $\underline{tr}$ such that $\underline{tr}\langle \alpha, n\rangle \Vdash p_n$. ☐

Theorem 5.21. *If X is a fibrant assembly, then there is a recursive function $\underline{\mathsf{con}}$ such that if $\alpha \Vdash q : I_n \to X$, then $\underline{\mathsf{con}}\langle n, \alpha, m\rangle \Vdash q(0), q(m-1), q(m)$ for $m > 0$ and $\underline{\mathsf{con}}\langle n, \alpha, 0\rangle \Vdash q(0)$.*

Proof. First, define a family of assemblies $\mathbf{N}_n$ for each $n \in \omega$, $|\mathbf{N}_n| = \{0, \dots, n\}$, $E_{\mathbf{N}_n}(i) = \{i\}$.

Then consider a commutative diagram

$$\begin{array}{ccc}
(\{0\} \times I \times (\Sigma_{n:\omega} X^{I_n} \times \mathbf{N}_n)) \cup (I \times (1+1) \times (\Sigma_{n:\omega} X^{I_n} \times \mathbf{N}_n)) & \xrightarrow{[a,b,c]} & X \\
\Big\downarrow{\scriptstyle \delta_0 \hat{\otimes} ([\delta_0,\delta_1]\times \mathrm{id})} & & \Big\downarrow \\
I \times I \times (\Sigma_{n:\omega} X^{I_n} \times \mathbf{N}_n) & \longrightarrow & 1
\end{array}$$

where the functions a, b, c are defined below (the last row contains realizers for the functions)

$$\begin{array}{ccc}
a(0,0,(-,q,m)) = q(m) & b(0,0,(-,q,m)) = q(m) & c(0,1,(-,q,m)) = q(0) \\
a(0,1,(-,q,m)) = q(0) & b(1,0,(-,q,m)) = q(m-1) & c(1,1,(-,q,m)) = q(0) \\
\lambda\langle -,i,-,\alpha,m\rangle.\underline{tr}\langle \alpha, m\rangle & \lambda\langle i,-,-,\alpha,m\rangle.\alpha\cdot m & \lambda\langle i,-,-,\alpha,m\rangle.\alpha\cdot 0
\end{array}$$

One can view the map $[a,b,c]$ as an "open box"

$$\begin{array}{ccc}
q(0) & = & q(0) \\
{\scriptstyle \underline{tr}(\alpha,m)} \Big\uparrow & & \\
q(m) & \xrightarrow[\alpha\cdot m]{} & q(m-1)
\end{array}$$

Since X is fibrant, we have a solution for the lifting problem above, which is a filler for the "open box". Effectively we have a map $con : I \times I \times (\Sigma_{n\in\omega} X^{I_n} \times \mathbf{N}_n)$, such that

$$con(0,0,(n,q,m)) = q(m)$$
$$con(0,1,(n,q,m)) = q(0)$$
$$con(1,0,(n,q,m)) = q(m-1)$$

for $q : I_n \to X$, $m \leq n$. Suppose *con* is tracked by φ. Since $1 \Vdash_I 0, 1$, we have

$$\varphi\langle 1, 1, \langle n, \alpha, m\rangle\rangle \Vdash con(0, 0, (n, q, m)), con(0, 1, (n, q, m)), con(1, 0, (n, q, m)) = q(0), q(m), q(m-1)$$

Thus we can define

$$\underline{\mathsf{con}} := \lambda x. \begin{cases} x = 0 & \to \mathsf{p}_2 x \cdot 0 \\ x \neq 0 & \to \varphi\langle 1, 1, x\rangle \end{cases}$$

which satisfies
$\underline{\mathsf{con}}\langle n, \alpha, m\rangle \Vdash q(0), q(m-1), q(m)$ for $m > 0$ and $\underline{\mathsf{con}}\langle n, \alpha, 0\rangle \Vdash q(0)$. $\square$

Theorem 5.22. *For a fibrant assembly X and maps $p, q : I_n \to X$ such that $p(0) = p(1) = \cdots = p(n-1) = q(0)$ and $p(n) = q(n)$, there is a path $I \to X^{I_n}$ connecting the two maps.*

Proof. Consider a realizer α for map $q : I_n \to X$. Then p and q share a realizer. Specifically,

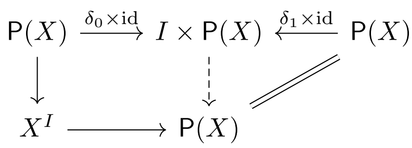
$$\lambda x.\underline{\mathsf{con}}\langle n, \alpha, x\rangle$$

as $\underline{\mathsf{con}}\langle n, \alpha, 0\rangle \Vdash q(0) = p(0)$; $\underline{\mathsf{con}}\langle n, \alpha, i\rangle \Vdash q(0) = p(i-1) = p(i), q(i-1), q(i)$ for $0 < i < n$ and $\underline{\mathsf{con}}\langle n, \alpha, n\rangle \Vdash q(0) = p(n-1), q(n) = p(n), q(n-1)$. $\square$

Theorem 5.23. *If X is fibrant assembly, then $\mathsf{P}(X)$ is homotopy equivalent to X^I.*

Proof. Given a path $p : I \to X$, we map it to $[(1, p)] \in \mathsf{P}(X)$. We shall show that this assignment $p \mapsto [(1, p)]$ has a homotopy inverse. We map an n-path (n, q) to a path q' with $q'(0) = q(0)$ and $q'(1) = q(n)$, q' is tracked by $\lambda x.\underline{tr}\langle \alpha, n\rangle$, where $\alpha \Vdash q$. This assignment does not depend on the representative of an equivalence in $\mathsf{P}(X)$ and thus defines a morphism $\mathsf{P}(X) \to X^I$.

We shall show the homotopy



Note that $[(1, p)] = [(n, p')]$ with $p'(0) = p'(1) = \ldots p'(n-1) = p(0)$ and $p'(n) = p(1)$. Then the map $I \times \mathsf{P}(X) \to \mathsf{P}(X)$ is tracked by $\lambda\langle -, n, \alpha\rangle.\lambda x.\underline{\mathsf{con}}\langle n, \alpha, x\rangle$.

$\square$

Notes

In this chapter we have applied the results obtained in chapter 3 to the effective topos $\mathcal{E}ff$. Starting with the interval object $I = \nabla 2$, we obtain a model category structure on $\mathcal{E}ff_f$, in which cofibrations are exactly monomorphisms. Accordingly, acyclic fibrations are maps that have the right lifting property against monomorphisms, and trivially fibrant objects are exactly the injective objects. We attempted to bring the familiar language of effective topos in describing this model structure. We showed that uniform objects are related to contractible object in the sense that trivially fibrant objects are exactly uniform objects that are fibrant. There are, however, uniform objects that are not contractible (and, therefore, not fibrant). In addition, if we restrict our attention to assemblies, then we can characterize trivial fibrations as uniform epimorphisms.

After that we discussed discrete objects and showed that they can be viewed as "discrete" in topological sense as well. Discrete objects happen to be fibrant, but not all discrete maps turned out to be fibrations. However, discrete maps with a discrete base are fibrations. It is unclear if we can precisely describe fibrant discrete maps.

We recalled the notion of a discrete reflection and showed that, when restricted to fibrant assemblies, the discrete reflection is a homotopy equivalence. From that we can derive a description of the homotopy category of assemblies as the category of modest sets.

In the next chapter we will see how this model structure gives rise to an interpretation of dependent type theory in $\mathcal{E}ff_f$.

Chapter 6

A model of type theory in the effective topos

This section is devoted to the exposition of the model of type theory in $\mathcal{E}ff_f$, arising from the Quillen model structure, in which types are interpreted as fibrant objects and type families are interpreted as fibrations. In the first part of this chapter, which does not contains any original results or contributions, we recall the general theory of categorical models of type theory. In section 6.1.1 we describe the underlying tool for modeling dependent type theory in a category – a category with families. All the type formers, such as Id or Π and Σ are interpreted on top of that structure. The support for the identity types in our model comes from the Quillen model structure, in a manner which we describe in section 6.1.2. On the other hand, the interpretation of Π and Σ types is obtained from the locally Cartesian structure of a topos, using the so called Frobenius condition, which we examine in section 6.1.3.

Then, in the second part we examine the aforementioned constructions in $\mathcal{E}ff_f$. We show that the model structure on $\mathcal{E}ff_f$ satisfy the Frobenius condition, and, thus, the model of type theory supports Π-types. We also show that the collection of types in $\mathcal{E}ff_f$ contains all finite types and is closed under product, sum, and Σ. We also explain how our model validates functional extensionality.

6.1 Categorical models of type theory

In this section we describe categorical semantics of type theory that we are going to use. First we will recall the "low level" semantics in terms of categories with attributes, and then we present more abstract homotopical semantics based on model categories.

6.1.1 Categories with attributes

In this section we will present a definition of a *category with attributes* (CwA), also known as *type-categories*, see Pitts [31, Section 6]. There is a wide variety of basic semantics for dependent type theory: categories with families of Dybjer [15], contextual categories of Cartmell [9, 39], to name a few. All those notions are essentially equivalent, and for our purposes they are supplemental. For more details see [20] and reference therein.

A *category with attributes* is a (small) category $\mathcal{C}$ with a terminal object, together with

- A functor $\mathrm{Ty} : \mathcal{C}^{op} \to \mathbf{Set}$. We write $(-)[f]$ for $\mathrm{Ty}(f) : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Delta)$ for $f : \Delta \to \Gamma$.
- For each $\Gamma \in \mathcal{C}$ and $T \in \mathrm{Ty}(\Gamma)$, an object $\Gamma.T$ (called "context extension") with a "projection map" $\mathsf{p}_{\Gamma.T} : \Gamma.T \to \Gamma$.

- For each $\Gamma \in \mathcal{C}$ and $T \in \mathrm{Ty}(\Gamma)$, and for each $f : \Delta \to \Gamma$, an arrow $q(f,T)$ making the following square a pullback:

$$\begin{array}{ccc} \Delta.T[f] & \xrightarrow{q(f,T)} & \Gamma.T \\ {\scriptstyle \mathsf{p}_{\Delta.T}}\downarrow & & \downarrow{\scriptstyle \mathsf{p}_{\Gamma.T}} \\ \Delta & \xrightarrow{\quad f \quad} & \Gamma \end{array}$$

We also require q is *functorial*, i.e. $q(\mathrm{id}_\Gamma, T) = \mathrm{id}_{\Gamma.T}$ and $q(g \circ f, T) = q(g,T) \circ q(f, T[g])$.

When context is understood, we write p_T for $\mathsf{p}_{\Gamma.T}$.

Loosely speaking, the category $\mathcal{C}$ is treated as a category of contexts and substitutions, types in a context Γ as elements of of $\mathrm{Ty}(\Gamma)$, and terms of types A as sections $\Gamma \to \Gamma.A$ to the projection map p_A. Substitution in types is interpreted as pullbacks. If $f : \Gamma \to \Delta$ is a substitution and $a : A$ is a term in context Δ (i.e. a section of p_A), then the substituted term $a[f] : A[f]$ is interpreted as a map $\langle \mathrm{id}_\Gamma, q(f,A) \rangle : \Gamma \to \Gamma.A$.

Types in a CwA. In order to interpret additional type formers (such as Π, Σ, N), we have to put additional conditions on a CwA. We encode type formation rules and term construction rules as additional algebraic structure on the CwA (in particular, we want morphisms of CwA to preserve them). And the definitional equality rules for types and terms are converted to axioms for those algebraic constructions. For instance, closure under Π type means the following.

Definition 6.1. A CwA $\mathcal{C}$ is said to have *dependent products* (or Π-types), if for each $\Gamma \in \mathcal{C}$, and each $X \in \mathrm{Ty}(\Gamma)$, $E \in \mathrm{Ty}(\Gamma.X)$,

1. There is a type $\Pi X E \in \mathrm{Ty}(\Gamma)$;
2. For each term $M : \Gamma.X \to \Gamma.X.E$ there is a term $\lambda(M) : \Gamma \to \Gamma.\Pi X E$;
3. There is a map $app_{X,E} : \Gamma.X.(\Pi X E)[\mathsf{p}_X] \to \Gamma.X.E$ such that $\mathsf{p}_E \circ app_{X,E} = \mathsf{p}_{(\Pi X E)[\mathsf{p}_X]}$;
4. For each term $M : \Gamma.X \to \Gamma.X.E$, the following equality holds:
$$app_{X,E} \circ \lambda(M)[\mathsf{p}_X] = M$$
5. The type former $\Pi X E$ and the elimination map $app_{X,E}$ are stable under substitution.

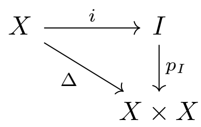
One can notice that those conditions correspond to type formation, term introduction, and elimination rules, respectively. Those conditions can be read off the syntactic rules for Π-types. Similarly we can write down categorical rules for Σ-types and other type formers. To see the constructions for other type formers, and to see the formal definition of what it means to be "stable under substitution", one can consult [20].

6.1.2 Homotopy theoretic models

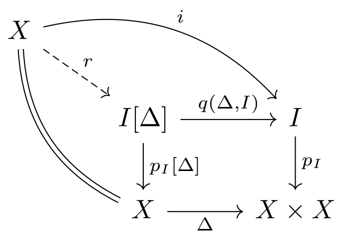
The novel connection between homotopy theory and type theory was highlighted in the work of Awodey and Warren [2], in which they constructed a model of identity types in a Quillen model category.

Identity types in model categories. The interpretation of dependent type theory in a model category is performed with the slogan "Fibrations are types". Specifically, one can build a model of type theory on a model category $\mathcal{C}$. For this, put $\mathrm{Ty}(\Gamma) = \{p : X \to \Gamma \mid p \text{ is a fibration } \}$. We also fix a choice of a pullback square for each substitution and a fibration. The context extension operation is interpreted simply as follows. If $p : X \to \Gamma$ is a fibration, then $\Gamma.p = \Gamma.X = X$. In order to interpret identity types we recall the following notion.

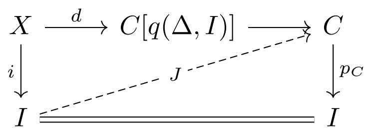
Definition 6.2. Given an object $X \in \mathcal{C}$ in a model category, a *very good path object* for X is a factorisation of the diagonal map $\Delta_X = \langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \to X \times X$ as an acyclic cofibration followed by a fibration.



Intuitively, if $X \xrightarrow{i} I \xrightarrow{p_I} X \times X$ is a very good path object for X, then the fibration $I \to X \times X$ corresponds to the judgment $[x : X, y : X \vdash \mathrm{Id}_X(x, y)]$ for a fibrant object X. The arrow $X \to I$ corresponds to a term $[x : X \vdash r(x) : \mathrm{Id}_X(x, x)]$. Specifically, the type $[x : X \vdash \mathrm{Id}_X(x, x)]$ corresponds to the pullback of p_I along Δ. Then the term $r(x)$ (section) in the context X is obtained from the universal property of the pullback:



Then, the identity elimination rule is interpreted as follows. Given a type $C(p, x, y) \in \mathrm{Ty}(X.X.\mathrm{Id}) = \mathrm{Ty}(I)$ and a term $x : X \vdash d : C(r(x), x, x)$ (as a section $d : X \to C[q(\Delta, I)]$), we can form a commutative diagram, as below.



which has a filler J, because p_C is a fibration corresponding to the type C and i is a trivial cofibration. The filler J represents a term $[a, b : A, p : \mathrm{Id}_A(a, b) \vdash J_C(d, a, b, p) : C(p, a, b)]$. The commutativity of the upper triangle models the computational rule for identity elimination.

By axioms of a model category such a factorisation always exists. To model identity types in a context Γ, one would then use a model category structure on $\mathcal{C}/\Gamma$. The axioms only guarantee the *existence* of path objects, they do not provide a specific "coherence" choice of a path object for each (fibrant) object X. We will sidestep those issues in this thesis, but we will note that there is a number of possible solutions. Warren in his theses [42] uses the Bénabou construction to obtain a "split model" from a split codomain fibration. Van Den Berg and Garner [6, 12] use *cloven factorisation systems*, where each fibration comes with a choice of fillers; it is also possible to utilize *algebraic weak factorisation systems* [7] for the same purpose.

6.1.3 Locally cartesian closed categories and Frobenius condition

It was first shown in the seminal paper of Seely [37] that a *locally cartesian closed category* (LCCC) has enough structure to interpret dependent type theory. Under Seely's interpretation, an LCCC $\mathcal{C}$ itself is seen as a "category of contexts and substitutions", and for a context $X \in \mathcal{C}$, the collection of types in the context X is the slice category $\mathcal{C}/X$[1], and substitution is given by the pullback functor $f^* : \mathcal{C}/Y \to \mathcal{C}/X$ for $f : X \to Y$. The extended context $X.A$ for a type $A \to X$ is given by A itself. That makes $\mathcal{C}$ a CwA. Furthermore, $\mathcal{C}$ supports Π and Σ types via the locally cartesian structure. Suppose $X \in \mathcal{C}$ and $A \xrightarrow{\mathsf{p}_A} X$ is a type over X, and $B \xrightarrow{\mathsf{p}_B} A \to X$ is a type over A. Then,

- The dependent product ΠAB is the domain of $\Pi_{\mathsf{p}_A}(\mathsf{p}_B)$.
- The application $app_{A,B}$ is given by the counit of the $\mathsf{p}_A \dashv \Pi_{\mathsf{p}_A}$ adjunction at p_B.
- Suppose that M is a term of type B in context A, i.e. $p_B \circ M = \mathrm{id}_A$. Since $p_A^*(\mathrm{id}_X) = \mathrm{id}_A$, the term M can be seen as a map $\mathsf{p}_A^*(\mathrm{id}_X) \to \mathsf{p}_B$ in $\mathcal{C}/A$. By transporting this map along the adjunction, we obtain $\lambda(M) : \mathrm{id}_X \to \Pi_{\mathsf{p}_A}(\mathsf{p}_B)$.

The computation rule follows from the adjunction properties. Once again, we sidestep the coherence issues; see [19, 11] for details.

Frobenius condition. But what happens if we want to combine the homotopical semantics of the identity types with the LCCC semantics of Π and Σ types? Suppose that we have a model category $\mathcal{C}$ with a choice of very good path objects, and in addition $\mathcal{C}$ is locally cartesian closed. We would like to interpret Π-types and Σ-types in the model induced by the model category structure. For that we must require the following:

1. If $p : X \to \Delta$ and $f : \Delta \to \Gamma$ are fibrations, then $\Pi_f(p)$ is a fibration as well.
2. If $p : X \to \Delta$ and $f : \Delta \to \Gamma$ are fibrations, then $\Sigma_f(p)$ is a fibration as well.

The case for Σ-types is perhaps not as important as that for Π-types. In a locally cartesian closed category $\Sigma_f(p) = f \circ p$ is a fibration as a composition of two fibrations.

The case for Π-types turns out to be equivalent to the so-called "Frobenius condition" or "Frobenius property".

A weak factorisation system (alternatively, a cloven factorisation system, and algebraic weak factorisation system, etc) is said to have a *Frobenius property* [6, 3.3.3(iv)], if the left class is stable under the pullbacks along maps from the right class. That is, given a pullback

$$\begin{array}{ccc} f^*(A) & \longrightarrow & A \\ {\scriptstyle \bar{i}}\downarrow & & \downarrow{\scriptstyle i} \\ \Delta & \xrightarrow{\ f\ } & \Gamma \end{array}$$

if i is a left map and f is a right map, then $\bar{i}$ is also a left map.

This property is useful for models of type theory without Π-types. To see this, consider the usual rule for Id-elimination

$$\frac{x : A, y : A, u : \mathrm{Id}_A(x,y) \vdash P(x,y,u)\ \text{type} \qquad x : A \vdash d(x) : P(x,x,r(x))}{x : A, y : A, p : \mathrm{Id}_A(x,y) \vdash J(d,x,y,p) : P(x,y,p)}$$

Under this formulation the type P can only depend on x, y and u. We want to allow P to depend on other arbitrary types and terms as well. Thus, we can reformulate the elimination rule as show in fig. 6.1.

In section 6.1.2 it was shown how to interpret the first elimination rule in a model category. To obtain the "weakened" rule, we must first pull back i along the projection map p_Δ, weakening the context of i

[1] Note that for *any* CwA $\mathcal{C}$ and for any object $A \in \mathcal{C}$, the collection of types $\mathrm{Ty}(A)$ can be organized into a full subcategory of $\mathcal{C}/A$; namely a subcategory that only contains projections $\mathsf{p}_T : A.T \to A$ as objects.

$$\frac{x : A, y : A, u : \mathrm{Id}_A(x, y), \Delta \vdash P(x, y, u) \text{ type} \qquad x : A, \Delta \vdash d(x) : P(x, x, r(x))}{x : A, y : A, p : \mathrm{Id}_A(x, y), \Delta \vdash J(d, x, y, p) : P(x, y, p)}$$

Figure 6.1: Modified Id elimination rule

$$\begin{array}{ccc} X.\Delta & \longrightarrow & X \\ {\scriptstyle \bar{i}}\downarrow & & \downarrow{\scriptstyle i} \\ I.\Delta & \xrightarrow{\mathsf{p}_\Delta} & I \end{array}$$

Then, if Frobenius property holds, $\bar{i}$ is an acyclic cofibration, and we can obtained the "weakened" elimination rule via the filler for the following diagram:

$$\begin{array}{ccc} X.\Delta & \longrightarrow & I.\Delta.P \\ {\scriptstyle \bar{i}}\downarrow & \overset{J}{\nearrow} & \downarrow{\scriptstyle \mathsf{p}_P} \\ I.\Delta & = & I.\Delta \end{array}$$

Frobenius and Π-types. In the presence of Π-types, the rule fig. 6.1 is derivable. For suppose $P(x, y, u, \delta)$ is a type in a context $x : A, y : A, u : \mathrm{Id}(x, y), \delta : \Delta$. Then we can form a type $\Pi_{\delta:\Delta} P(x, y, u, \delta)$ in a stronger context $x : A, y : A, u : \mathrm{Id}(x, y)$, with which we can apply the standard Id elimination rule.

Similarly, Frobenius property is equivalent to the condition that fibrations are closed under Π_p for every fibration p. Although this fact is widely known, to our knowledge the complete proof is not present in the literature. We state it as a proposition and put the proof in the appendix.

Proposition 6.3. *Suppose a locally cartesian closed category $\mathcal{C}$ has a structure of a model category. If $\mathcal{C}$ satisfies the Frobenius condition, then $\mathcal{C}$ supports Π-types.*

Proof. This is corollary A.2. □

6.2 Description of the model in $\mathcal{E}ff_f$

As we have seen in the previous section, the model category structure on $\mathcal{E}ff_f$ induces a model of type theory. In this section we would like to show that this model supports all the usual type formers of Martin-Löf type theory. We work in $\mathcal{E}ff_f$.

Identity types. For $\mathcal{E}ff_f$ to support identity types is, to give a choice of a nice path object for each fibrant object X. We can simply take the factorisation of $X \xrightarrow{\langle \mathrm{id}, \mathrm{id} \rangle} X \times X$ to be

$$X \xrightarrow{r} X^I \xrightarrow{\langle s,t \rangle} X \times X$$

The map $r : X \to X^I$ is an acyclic cofibration by remark 2.24, and the map $\langle s, t\rangle : X^I \to X \times X$ is a fibration by proposition 2.5.

Exponents and functional extensionality. The fact that fibrant objects are closed under exponentiation follows from the following proposition, taking $X = 1$.

Proposition 6.4. *If $p : E \to X$ is a fibration and Y is an object, then $p^Y : E^Y \to X^Y$ is a fibration.*

Proof. Since we take cofibrations to be all monomorphisms, this is just remark 2.7. □

By functional extensionality we mean a map $e : \mathrm{Id}(X)^Y \to \mathrm{Id}(X^Y)$, such that

$$(\langle s, t\rangle)^Y = \langle s, t\rangle \circ e$$

where $\mathrm{Id}(X)$ denotes the identity type for X with the projection $\langle s, t\rangle : \mathrm{Id}(X) \to X \times X$. Thus $\mathrm{Id}(X)^Y$ is an object of homotopical maps $X \to Y$, and $\mathrm{Id}(X^Y)$ is a path object of maps $X \to Y$. In Van Oosten's model the identity type $\mathrm{Id}(X)$ is interpreted as $\mathsf{P}(X)$, and does not support functional extensionality (see example 5.19). Because in our model the identity type is interpreted as $\mathrm{Id}(X) = X^I$, we get the functional extensionality automatically, as in any Cartesian closed category: $(X^I)^Y \simeq X^{I\times Y} \simeq (X^Y)^I$.

Product types. The following easy proposition implies that $\mathcal{E}ff_f$ supports product types.

Proposition 6.5. *If $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ are fibrations, then so is $p_1 \times_X p_2 : E_1 \times_X E_2 \to X$, which is obtained from a pullback*

$$\begin{array}{ccc} E_1 \times_X E_2 & \xrightarrow{\pi_2} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ E_1 & \xrightarrow[p_1]{} & X \end{array} \qquad p_1 \times_X p_2 := p_2 \circ \pi_2 = p_1 \circ \pi_1$$

Proof. By proposition 2.8, both π_1 and π_2 are fibrations. The result follows from the fact that the composition of fibrations is a fibration. □

Sum types. In order to show that $\mathcal{E}ff_f$ support sum types, we first state the following auxiliary proposition.

Proposition 6.6. *If $p_1 : E_1 \to X_1$ and $p_2 : E_2 \to X_2$ are fibrations, then so is $p_1 + p_2 : E_1 + E_2 \to X_1 + X_2$.*

Proof. Virtually the same as in proposition 2.10 □

Noting that the assembly $2 = \{0, 1\}$ with the realizability relation $E_2(i) = \{i\}$ is a modest set, we can show that types in $\mathcal{E}ff_f$ are closed under sums.

Proposition 6.7. *If $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ are fibrations, then so is $[p_1, p_2] : E_1 + E_2 \to X$.*

Proof. The arrow $[p_1, p_2]$ can be written as a composition $E_1 + E_2 \xrightarrow{p_1+p_2} X + X \xrightarrow{[\mathrm{id},\mathrm{id}]}$ (this essentially amounts to encoding $[x : X \vdash E_1(x) + E_2(x)$ type$]$ as $[x : X \vdash \Sigma_{i:2}E_i(x)$ type$]$ in type theory). The first composite $p_1 + p_2$ is a fibration by proposition 6.6. The second composite is isomorphic to $X \times 2 \xrightarrow{\pi_1} X$. Since fibrations are closed under composition, it suffices to show that $X \times 2 \xrightarrow{\pi_1} X$ is a fibration. By remark 2.11 this holds if $2 = 1 + 1$ is fibrant. One can note that 2 is in fact a modest set in $\mathcal{E}ff$ and thus is fibrant. □

Finite types and natural numbers. A standard Martin-Löf type theory contains all finite types; we can obtain them starting from $\emptyset$ and 1 using sums.

Proposition 6.8. *Both $\emptyset$ and 1 are fibrant.*

Proof. The terminal object 1 is always fibrant, as $1 \to 1$ is an isomorphism.

To show that $\emptyset$ is fibrant, suppose that we have to extend a map $(I\times A)\cup(\{0\}\times B) \to \emptyset$ along the inclusion $(I \times A) \cup (\{0\} \times B) \xrightarrow{\delta_0 \hat{\otimes} u} I \times B$. Since there is a map $(I \times A) \cup (\{0\} \times B) \to \emptyset$, the object $(I \times A) \cup (\{0\} \times B)$ is isomorphic to $\emptyset$; hence $\{0\} \times B \simeq \emptyset$ and $I \times B \simeq \emptyset$. □

Proposition 6.9 (Example 5.15)**.** *The natural numbers object $\mathbf{N}$ is fibrant.*

Finally, we move on to the most important type formers (other than the identity type) – Π and Σ.

6.2.1 Π and Σ types

Since $\mathcal{E}ff$ is a topos, we set up an interpretation of Π and Σ types in $\mathcal{E}ff$ as described in section 6.1.3. To show that $\mathcal{E}ff$ supports Σ and Π types we must show that the class of fibrations is closed under Σ_p and Π_p for every $p \in \mathcal{F}ib$.

Σ-types. If $f : A \to X$ is an arrow in $\mathcal{E}ff$, then $\Sigma_f : \mathcal{E}ff/A \to \mathcal{E}ff/X$ is defined on objects as $p : E \to A \mapsto f \circ p : E \to X$, and on arrows as $g : p \to q \mapsto g : (f \circ p) \to (f \circ q)$.

Because fibrations are closed under composition, we immediately have the following proposition.

Proposition 6.10. *$\mathcal{E}ff_f$ supports Σ-types; that is, if $f : A \to X$ and $p : E \to A$ are fibrations, then so is $\Sigma_f(p) : E \to X$.*

Π-types. The description of Π-types in $\mathcal{E}ff$ is slightly more involved. Luckily for us, we don't have to dwell into the details, as we can utilize proposition 6.3. In other words it suffices to check that monomorphisms that are homotopy equivalences are stable under pullbacks along fibrations. Monomorphisms are always stable under pullbacks, so the condition is equivalent to the right-properness of the model category structure. However, we get right-properness "for free", from the following proposition.

Proposition 6.11 ([18, Proposition 13.1.2])**.** *If $\mathcal{M}$ is a model category, then*

1. *Every pushout of a weak equivalence between cofibrant objects along a cofibration is a weak equivalence;*
2. *Every pullback of a weak equivalence between fibrant objects along a fibration is a weak equivalence.*

Since every object in our model category is fibrant and cofibrant, we can conclude that the resulting model structure is left- and right-proper. Because cofibrations and weak equivalences are stable under pullbacks along fibrations, we can conclude that our model supports Π-types.

Functional extensionality for Π-types. By a type theoretic argument [40, Proposition 4.9.5], functional extensionality for dependent functions is equivalent to the so called *weak functional extensionality*, which says that contractible maps (trivial fibrations) are stable under Π along fibrations. That is, if $X \xrightarrow{f} \Gamma$ is a fibration (representing type X in context Γ), and a fibration $E \xrightarrow{p} X$ (representing type $E(x)$ in context $[\Gamma, x : X]$) is trivial, then the fibration $\Pi_f(p) : \Pi_X E \to \Gamma$ is also trivial. Type theoretically, this states that if every fiber $E(x)$ is contractible, then the compound type $\Pi_{x:X} E(x)$ is contractible as well.

By corollary A.3 this is equivalent to the statement that cofibrations are stable under pullbacks along fibrations. Since in our model cofibrations are exactly monomorphisms and monomorphisms are stable under pullbacks, we can conclude that $\mathcal{E}ff_f$ supports weak functional extensionality, and, therefore, functional extensionality for Π-types.

Notes

In this chapter we have seen how to define a general categorical model of dependent type theory. Our notion of choice – Categories with Attributes – provides only the bare support for interpreting type theory. In order to interpret identity types and Π-types we appeal to a closed model category structure. Under this homotopical interpretation, types are taken to be fibrations. The factorisation of the diagonal map $X \to X \times X$ as an acyclic cofibration followed by a fibration provides us with an identity type for X. Furthermore, if we want to interpret Π types as they are usually interpreted in a locally Cartesian closed category, then we have to require that $\Pi_f(p)$ is a fibration whenever both f and p are.

As it turns out, this condition does hold in the effective topos (or, rather, in our model category structure on $\mathcal{E}ff_f$), from which can conclude that our model supports identity types and Π-types. We also show that our model supports all finite types and validates functional extensionality for exponents and for Π-types.

Conclusions and future work

In the first part of this thesis we have presented the construction of a model category on a full subcategory of any elementary topos, starting from the interval object I with connections. In the resulting model category every object is fibrant, cofibrations are monomorphisms and weak equivalences are homotopy equivalences with regard to the interval I. Because cofibrations are monomorphisms, every object is cofibrant and the model structure is proper.

We have applied this construction to the specific case of the effective topos $\mathcal{E}ff$, taking the interval object to be an assembly $\nabla 2$. This results in the notion of homotopy in $\mathcal{E}ff$ in which "discrete spaces", i.e. objects X such that any map $I \to X$ factors through 1, are exactly discrete objects from the theory of realizability toposes. In this model structure, contractible objects correspond to fibrant uniform objects. Determining whether an arbitrary object of the effective topos is fibrant is still rather hard, and, consequently, it is hard to give a concrete description of the homotopy category $Ho(\mathcal{E}ff_f)$. However, using the discrete reflection we manged to show that the homotopy category of fibrant assemblies $Ho(Asm_f)$ is equivalent to the category of modest sets.

The model category structure on $\mathcal{E}ff_f$ gives rise to an interpretation of dependent type theory in which the identity type of an object X is interpreted as a function space X^I. This interpretation supports full Martin-Löf type theory with $\Pi, \Sigma, \mathbf{N}, 0, 1, \times, +$. Functional extensionality is valid in this model both for simple exponents and for dependent products.

We would like to highlight some possible directions of future work in this area.

1. *Extending the correspondence developed in section 5.2.1.* As of now, it is unclear if there is a correspondence between trivial fibrations and uniform epimorphisms that are fibrations, in the most general case. One would hope that the situation should be analogous to contractible objects and uniform fibrant objects.

2. *Higher inductive types.* In the aforementioned section we have also presented a "circle" S, as a counterexample to the claim that uniform objects are contractible. It is the case that S has non-trivial paths, however, S cannot be fibrant and therefore is not a type. Is it possible to construct a fibrant "circle"? Is it possible to construct any other higher inductive types?

3. *Fibrant replacement & extension of the model structure.* One way to turn S from the previous point into a fibrant object is to use a fibrant replacement monad: factor the map $S \to 1$ as an acyclic factorisation followed by a fibration. This will result in a fibrant object S' that is homotopic to S. However, in order to use the fibrant replacement, one would have to extend the model structure to the whole of the effective topos, and not restrict oneself to fibrant objects. For this, probably, one would need to change the definition of a homotopy equivalence. Van Oosten's path object construction [28] might come in handy, since it induces a "well-behaved" notion of homotopy that (almost) coincides with our definition of homotopy on fibrant objects.

4. *Fibrant universe.* Yet another peculiarity of the effective topos, is that it contains an internal small complete category of modest sets [21]. An existence of a small complete category that is not a preorder is inconsistent with classical logic and the

logic of Grothendieck topoi. This internal category is represented by the *universal discrete map* (meaning that every discrete map can be obtained by pulling the universal map back), and it is used as a type-theoretic universe for impredicative type systems. The problem is that the universal discrete map cannot be fibrant, as witnessed by example 5.14. The natural question to ask is then the following: can we find a subobject of the universal discrete map, that is universal for discrete maps that are also fibrations?

5. *Relating our approach to other constructions.* It would also be interesting to see how does our approach leverage when compared to other construction based on interval-like objects. For instance, in Warren [42] devised a method of constructing models of type theory starting from the interval object, but he considers a different definition of a fibration – a Hurewicz-style fibrations. In the upcoming work [30] Orton & Pitts construct a model of *cubical type theory* in a topos with an interval object, but they assume that the topos comes with a full internal subtopos serving as a universe for types.

6. *Coherence issues.* We have politely sidestepped the coherence issues in this work, due to somewhat laborious nature of the task and due to time constraints. However, it seems like a fairly natural improvement. It is possible to resolve the coherence issues by considering algebraic counterparts of the homotopy-theoretic notions considered in this thesis, such as algebraic weak factorisation systems and algebraic model structures. It also seems that the approach presented in this thesis might be very susceptible to algebraization, given that the original work of Gambino & Sattler [17] has the algebraization built-in.

Appendix A

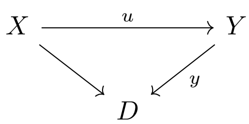
Frobenius condition and Π-types

In this appendix we present a proof of a folklore theorem which states that the Frobenius condition (in the sense of section 6.1.3) is equivalent to the preservation of fibrations by Π's. First we prove a more general result and the derive the required proposition as a corollary.

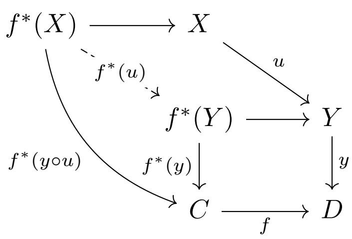
Theorem A.1. *Suppose $\mathcal{C}$ is a locally cartesian closed category, and $\mathcal{L}, \mathcal{R}$ are two classes of maps in $\mathcal{C}$ such that $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\pitchfork}$, and $\mathcal{F}$ is a class of morphisms in $\mathcal{C}$ that is closed under pullbacks . Then the following two conditions are equivalent:*

1. *$\mathcal{L}$ is closed under pullbacks along maps from $\mathcal{F}$.*
2. *$\mathcal{R}$ is closed under Π_f for every $f \in \mathcal{F}$.*

Proof. Suppose that condition (1) holds. That is, if $f : C \to D \in \mathcal{F}$ and $u : Y \to D \in \mathcal{L}$, then $f^*(u)$ is in $\mathcal{L}$ as well. Thus, the pullback functor $f^* : \mathcal{C}/D \to \mathcal{C}/C$ sends $\mathcal{L}$-maps, as objects of $\mathcal{C}/D$, to $\mathcal{L}$-maps, as objects of $\mathcal{C}/C$. First, we will show that f^* also sends $\mathcal{L}$-maps as morphisms in $\mathcal{C}/D$ to $\mathcal{L}$-maps as morphisms of $\mathcal{C}/C$. Consider a map $u : y \circ u \to y$ in $\mathcal{C}/D$:



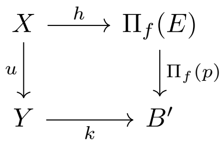
The action of f^* on this morphism is a morphism $f^*(u) : f^*(y \circ u) \to f^*(y)$ which is obtained from the universal property of the pullback:



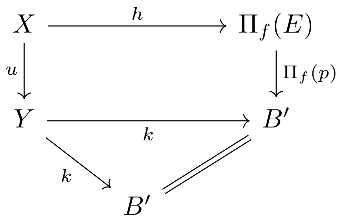
The outer square is a pullback of $y \circ u$ along f, and the lower inner square is a pullback of y along f. Hence, by the pullback lemma, the upper inner square is a pullback as well.

The map $f^*(Y) \to Y$ is a pullback of f, hence it is also in $\mathcal{F}$ by the assumption. Thus, by condition (1), if u is in $\mathcal{L}$, then $f^*(u)$ is also in $\mathcal{L}$.

Now suppose that $p : E \to B$ is in $\mathcal{R}$, $f : B \to B'$ is in $\mathcal{F}$. We wish to show that $\Pi_f(p) \in \mathcal{R}$; suppose that $u : X \to Y$ is in $\mathcal{L}$ and we have a commutative diagram in $\mathcal{C}$:



We can view it as commutative diagram in $\mathcal{C}/B'$:



By proposition 1.8, the adjunction $\mathcal{C}/B' \underset{\Pi_f}{\overset{f^*}{\rightleftarrows}} \mathcal{C}/B$ ($\perp$) lifts to the adjunction between arrow categories. Thus, we can transport the diagram above to get a commutative square $f^*(u) \to p$. By the discussion above, $f^*(u)$ is in $\mathcal{L}$, and thus the square has a diagonal filler.

The proof in the other direction $(2) \to (1)$ is similar. □

Corollary A.2. *Suppose $\mathcal{C}$ is a locally cartesian closed category and $\mathcal{D} \hookrightarrow \mathcal{C}$ is a full subcategory of $\mathcal{C}$. Let $(\mathcal{W}, \mathcal{F}ib, \mathcal{C}of)$ be a model category structure on $\mathcal{D}$. Suppose also that $\mathcal{C}of \cap \mathcal{W}$ is closed under pullbacks along the maps from $\mathcal{F}ib$. If p, f are fibrations, then so is $\Pi_f(p)$.*

Proof. Apply theorem A.1 with $\mathcal{L} = (\mathcal{C}of \cap \mathcal{W})$, $\mathcal{R} = \mathcal{F}ib$ and $\mathcal{F} = \mathcal{F}ib$. □

Corollary A.3. *Suppose $\mathcal{C}$ is a locally cartesian closed category and $\mathcal{D} \hookrightarrow \mathcal{C}$ is a full subcategory of $\mathcal{C}$. Let $(\mathcal{W}, \mathcal{F}ib, \mathcal{C}of)$ be a model category structure on $\mathcal{D}$. Suppose also that $\mathcal{C}of$ is closed under pullbacks along the maps from $\mathcal{F}ib$. If f is a fibration and p is an acyclic fibration, then so is $\Pi_f(p)$.*

Proof. Apply theorem A.1 with $\mathcal{L} = \mathcal{C}of$, $\mathcal{R} = (\mathcal{F}ib \cap \mathcal{W})$ and $\mathcal{F} = \mathcal{F}ib$. □

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